Theoretical Error Bounds for the Value and Policy Function Iteration Algorithms: An Application for Recursive Dynamic Models with Inequality Constraints

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Abstract: This paper derives theoretical error bounds for the Policy and Value Function Iteration algorithms applied to Recursive Dynamic Models with continuous decision variables and inequality constraints. This paper proves two main theorems. The first one uses a recent result due to Santos and Rust (2004). The theorem extends the result by combining a feasible version of the Policy Function Iteration algorithm with the barrier method for a model with an arbitrary number of state and decision variables. This constitutes a significant difference with the original theorem, since it is no longer necessary to assume the interiority of the solutions. The Algorithm converges at a rate of 1.5 for a given grid size. The second theorem, for problems with only one continuous endogenous state variable, uses a feasible version of the Value Function Iteration Algorithm, the barrier method and a cubic Variation Diminishing Spline Approximation. The algorithm converges at a linear rate, given the grid size. Finally, under a certain configuration of the parameters, the maximization problem in this last theorem is in the convex class, which can be solved in polynomial type complexity, and the policy function is first order differentiable. These last results enables the algorithm to avoid the Course of Dimensionality for the maximization problem in the Bellman Equation and the use of first order perturbation methods, thus, constitutes an extension of existing theorems that deals only with equality constraints (see Judd, 1994).

Keywords: error bounds, numerical dynamic programming, inequality constraints.

JEL Classification: C61, C63, C65

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1. Introduction

Dynamic economic models with occasionally binding inequality constraints have always been treated in a separate way in the literature of numerical methods. For instance, in a recent survey (Arouba, et. al, 2006) the authors compare several methods to solve recursive dynamic macroeconomic models, specifically the neoclassical growth model, assuming that the policy function belongs to the interior of the feasible region. While this assumption have no economic relevance in the case of boundary restrictions, it may be important when the occasionally binding constraints play a central role in the model. This is the case, for instance, in the Cash-in-Advance Model, in the irreversible investment model of growth and in Real Business Cycle Models with Heterogeneous agents and idiosyncratic liquidity or debt constraints.

From another perspective, these models almost always fall in the class of Markov Decision Process (MDP) with continuous state and decision variables. Then, they are affected by the course of dimensionality (Chow, et. al. 91), meaning that the worst case complexity of the problem grows exponentially with the number of states and controls. Further, inequality constraints causes the optimization problem to suffer the combinatorial difficulty of nonlinear programming (Nocedal, Ch. 15), meaning that the number of choices in an active set strategy grows at a rate of 2^m, where m is the number of inequality constraints. Besides, the possibility of a binding constraint does not permit the differentiability of the value function, at least without imposing stronger restrictions on the problem (see, for instance, Santos and Rincon Zapatero, 2009). So, it is not possible to use neither Euler Equation Methods (Rust, 1994) nor Perturbation Methods (Judd, 1998, Ch. 13 to 15) without further structure. All these features had led the numerical optimization and numerical dynamic programming literature to develop algorithms that treat all these problems carefully, leading to a new promising branch in the literature.

In a recent paper, Christiano and Fischer (2000) performed an exhaustive numerical test on a set of algorithms that they considered appropriate to deal with the mentioned problems. They developed a series of smooth approximation algorithms for the irreversible investment neoclassical growth model with only one endogenous continuous and one exogenous discrete state variables. They chose a version of the Parametrized Expectations Algorithm (PEA, Marcet and Marshall, 1994) because it was the fastest method with a relatively high accuracy. Even though these results are useful from a practitioner's point of view, theoretical error bounds have not been derived yet. Besides, unless the true function lies within the subspace spanned by the basis functions used in the algorithm, smooth approximation methods cannot provide arbitrarily accurate approximations. As a result, to obtain a benchmark solution, Christiano and Fischer use a discrete approximation method: the Value Function Iteration (VFI), taking into account the inequality constraint with a polynomial approximation with an extremely fine grid. Available results from Santos (1998) allow to think that the VFI could provide an arbitrarily accurate solution for the unconstrained o even equality constrained case. However, it is not clear how the approximation of the inequality constraint would affect the performance of the method, especially in terms of the accuracy. One of the purposes of this paper is to complement the results in Christiano and Fischer (2000), providing benchmark solutions for the inequality constrained case.

In general, the lack of a closed form in a model requires a benchmark algorithm, because it provides a set of numerical solutions within a desired level of accuracy. The typical algorithms used to serve this purpose were the VFI and Policy Function Iteration (PFI) because of existing theoretical error bounds for models with interior solutions. These results are due to Santos (1998) and Santos and Rust (2004). However, with the notable exception of the PEA[1], there have been no results on error bounds for MDP with continuous states and inequality constraints. This paper will prove the existence of such error bounds, complementing the Santos and Rust results.

This paper derives theoretical error bounds for computationally feasible globally convergent algorithms that solve models in the Euler Class (Rust, 1994, Ch. 2) with inequality constraints. To deal with this problem it is necessary to take care of three different error sources: the approximation error (APE), the fixed point error (FPE) and the maximization error (ME). The first one is the cost of making the algorithm tractable. As it is only possible to obtain a finite number of points of the true function, there is a need to interpolate them to attain a smooth result. Two different interpolating procedures are chosen: Piewise linear (PLA) and Variation Diminishing Spline (VDSA) Approximations. The FPE is due to the finiteness of number of iterations in an iterative discrete approximation algorithm, in this case the VFI and the PFI algorithms. Finally, the ME is derived from the numerical errors

[1] It must be mentioned that the assumptions made to derive the asymptotic convergence of the PEA seem to be too demanding for the mentioned context (see Rust, 1994, page 140).

involved in the maximization algorithms, as asymptotic convergence to the set of constrained maximizers is assured under weak assumptions. To handle inequality constraints in the maximization problem, this paper uses a Barrier Algorithm. Even though there are applications of the barrier method to accommodate inequality constraints in practice (see for instance, Preston and Roca, 2007), this paper constitutes the first attempt to use this method to derive asymptotic results in the literature of numerical dynamic programming.

With the mentioned tools, the paper states and proves two main theorems. Theorem 1, for a model with an arbitrary number of endogenous (continuous) states variables, using the PFI, the Barrier Method and PLA, shows that the FPE decreases at a rate of 1.5, as the number of iterations goes up, but increases with the grid size. These results are an extension of the ones obtained by Santos and Rust (2004), taking into account inequality constraints. Besides, it is shown that the APE decreases monotonically with the grid size. Finally, the paper proves that the ME can be bound by adjusting a parameter in the barrier method. Thus, in a sense, Theorem 1 constitutes an extension of the existing results as it allows to account for the three possible sources of error as a function of the parameters of the model. Note that a trade off arises between the FPE and the APE as the grid size becomes smaller. This last conclusion can be used to explain the collapse of the PFI found in the Christiano and Fischer paper and their preference for the VFI for an extremely fine grid. Moreover, with the help of Theorem 1, it is possible to measure this trade off, as it depends only in the deep parameters of the model, and to determine precisely whether to use the VFI or the PFI. Theorem 2, using the VFI, the Barrier Method and a cubic VDSA, for a model with only one endogenous continuous state variable, shows that FPE decreases at a linear rate, the APE decreases monotonically with the grid size and the ME can be controlled using the barrier method. Further, imposing some restrictions to the set of Lagrange multipliers and to the objective function, theorem 2 allows to derive asymptotic convergence even when the maximizers lie on the boundary of the feasible region. This last result is a significant improvement since it allows to deal with binding constraints (the Cash-in Advance or Irreversible Investment constraints), without interiority assumptions (usually required in the existing literature) and to obtain a differentiable path for the set of barrier maximizers, as well as an estimate of the Lagrange multiplier. Finally, for small values of the barrier's parameter, the results imply that the maximization problem is in the convex class, which, as it is well known, can be solved in polynomial time complexity using a gradient hill algorithm of the Newton-Quasi Newton Type (see

Nemirovsky and Yudin, 1985). The importance of this last result was highlighted in Judd (1994, Page 153). One of the most well kwon threats to the tractability of the VFI is the course of dimensionality in the maximization problem. The assumptions in Theorem 2 allow us to avoid it, leaving us with only the approximation problem due to a continuous decision variable. This result constitutes an extension of the one in Judd (1994), as it allows to deal with an arbitrary number of constraints, without imposing interiority assumptions. Finally, note that the barrier method implies that the feasible solutions are always interior, thus allowing for second order differentiability of the value function and first order differentiability of the policy function. This last result can easily be applied to derive first order perturbation (local) methods. However, it must be noted that the policy function is differentiable only up to the first order, and second order perturbation methods (see for instance Uribe, et. Al., 2004) are not well defined in the presence of inequality constraints. In the literature of real business cycle models with heterogeneous agents, the linearity of the saving function is often tested using second order perturbation methods (see for instance, Preston and Roca, 2007). Remark 2 of Theorem 2 states that the policy function is only first order differentiable, consequently the use of higher order approximations may be misleading.

The paper is organized as follows. Section 2 presents an abstract version of the theoretical model together with the algorithm's building blocks. This section presents the contributions of recent literature that are used as inputs for the two main theorems. Section 3 states the two main theorems and proves them. Section 4 concludes.

2. Preliminaries

2.1 The Model

Let $K \subset \mathbb{R}^n$ and $Z = \{z_i\}_{i=1}^{\bar{z}}$ with $z_i \in \mathbb{R}^l, i = 1,2 \dots, \bar{z}$ be the set of all possible endogenous and exogenous state variables respectively. Define $S = [KxZ]$ as the state space.

Note that it is assumed that $\#Z < \infty$. Further, suppose that the exogenous state variables are driven by a (finite state) first order Markov Process with transition matrix P.

The physical constraints of this economy are given by a correspondence, $\Gamma: S \to K$ and, as usual, $\beta \in (0,1)$, where β is the discount factor.

The following set of additional assumptions guarantee the existence of a recursive (functional) version of the original (sequential) problem, which is omitted for the sake of concreteness (see Stockey, Lucas and Prescott, Ch. 4, sections 4.1 and 4.2 for a related discussion).

Assumption 1: Γ, the feasibility correspondence, is continuous, compact and convex valued.

Assumption 2: Let u denote the return function. Assume then that, $u: KxKxZ \to R$, $u \in C^2$ in $int(\Omega_z)$, where Ω_z is the graph of $\Gamma(.;z)$ and $u(.;.;z)$ is strictly concave.

Assumption 3: for each $k_0, z_0 \in S$, the optimal solution $\{k_t, z_t\}_{t=1}^{\infty} \in int(\Omega_z)$.

Given that the optimization problem includes inequality constraints, section 3 drops this last assumption and introduce the barrier method instead.

These assumptions allow to define the Bellman equation, W. From standard arguments, W is well defined, continuous and strictly concave,

1. $$W(k_0, z_0) = \max_{k' \epsilon \Gamma(k_0,z_0)} \{u(k_0, k', z_0) + \beta \sum_{z' \epsilon Z} W(k', z') Q(z', z_0)\}$$

Where $Q(.,.)$ is the conditional transition probability, which is derived from P.

Note here the effects of the imposed structure on the stochastic process driving the exogenous state variables. This assumption is done only to keep the standard assumptions in the literature (Christiano and Fischer, 2000). Since the focus is on the effects of the inequality constraints on the maximization problem in equation 1, it makes no sense to deal also with the numerical integration issues that will arise with a continuous exogenous state.

The maximal elements in equation 1 are characterized by a policy function $k' = g(k,z)$, which it is known to be continuous. Note that $g(.,.)$ is a function only because of the strict concavity assumption on the return function and the shape preserving property of the Bellman Operator implicitly defined in equation 1. However, when dealing with a tractable version of numerical dynamic programming algorithms, the map $g(.,.)$ could perfectly be a correspondence instead of a function. This is because the shape preserving properties of the Bellman operator are no longer preserved under most of the interpolating procedures available in the numerical analysis literature.

The (non linear) operator in the Bellman equation, T_E, (see equation 2 below) could be used to define a recursive mapping from a Banach Space into itself. This stresses that the solution to the problem will be exact (E). Further, endowing the space of bounded continuous functions, $\mathcal{W}$, with the Sup-Norm, allows to obtain a complete metric space from T_E. The Blackwell Sufficient conditions guarantee the existence of unique solution to equation 1. Formally,

2. $$T_E(W)(k_0, z_0) = \max_{k' \epsilon \Gamma(k_0, z_0)} \{u(k_0, k', z_0) + \beta \sum_{z' \epsilon Z} W(k', z') Q(z', z_0)\}$$

With $\|T_E(W_1) - T_E(W_0)\| \leq \beta \|W_1 - W_0\| \; \forall \, W_1, W_0 \in \mathcal{W}$. Further, the fixed point in 2 satisfies:

3. $$\|W - T_E^n(W_0)\| \leq \beta^n \|W - W_0\|$$

Note that this simple structure guarantees the linear convergence (at a rate β) of the iterative (non tractable) algorithm implicit in 2. The non tractability comes from assuming an exact solution to 2 in $\mathcal{W}$, for each of the n iterations. Section 3 presents a tractable version of this algorithm kwon as the VFI.

2.2 Differentiability of the Value Function

The second order differentiability of the value function will be used to set bounds to the APE that comes from using piecewise linear interpolation.

Further, as regards the numerical aspects of the maximization problem in the Bellman equation, the existence of a well defined Hessian would allow for the use Gradient Hill Newton Methods, which are essential to avoid the ill conditioning problems that produces the barrier method (see Nocedal and Wright, 2006, Ch. 17).

For the sake of concreteness, this discussion does not consider the first order derivative of the value function and refer the reader to the well known literature (Stockey, Lucas and Prescott, Ch. 4 section 2). However two aspects must be highlighted: first-order differentiability requires a concave return and value functions and a set of interior solutions[2]. Second, in the presence of inequality constraints, there is no guarantee of the existence of such a set of solutions. These problems induce to introduce barrier methods, which prevent barrier maximizers from reaching the boundary of the feasible region.

Consider the theorem that guarantees the second order differentiability of the value function.

Theorem 2.1: Under Assumptions 1-3, $W \in C^2$ in $int(\Omega_z)$ and $D_{1,1}W(k,z)$ is bounded. In particular,

$$\left\|D_{1,1}W(k,z)\right\| \leq \left\|D_{1,1}u(k,g(k,z),z)\right\| \leq B$$

Where B is the supremum of the Hessian of the return function.

Proof: see Gallego (1993)

In particular it should be noted that Gallego (1993) assumes that the return function is α-concave[3], following the functional analysis literature. However on compact sets, every C^2 strictly concave function is α-concave (see Gallego Pag. 15). So, Assumption 1 and 2 guarantee the α-concavity of the return function. Besides, the author works in a deterministic environment. However, as noted by Santos (1998), the analysis is easily extended to the present framework (i.e. a discrete stochastic process).

The first order conditions that characterized the maximal elements in the Bellman equation, together with theorem 2.1 establish the differentiability of the policy function. In particular, an application of the implicit function theorem is sufficient to obtain that property (see Theorem 3.2 in Gallego, 1993).

[2] In the case of non interior solutions, we must require stronger assumptions (see Renthall, 2006 or Santos and Rincon Zapatero, 2009). As is well kwon from the convex analysis literature, a concave function is Lipschitz continuous in a point of the domain z if and only if it is bounded in a neighborhood of z (see Borwein and Lewis, 2000, Theorem 4.1). So in the boundary of the feasible region, even a concave function has no well defined derivative.

[3] A function $f: I \subset \mathbb{R}^N \rightarrow \mathbb{R}$ *and* $\alpha \in (0,1)$ is said to be α-concave if $f(tx) \geq t^{\alpha} f(x)$ for $\in I$ *and* $t \in (0,1)$.

Theorem 2.1 could be of particular importance, as Perturbation Methods can be used to approximate local solutions. In a recent paper Arouba et. al. (2006), state that it is not possible to use perturbations methods in the presence of inequality constraints (see Pag. 3). However, using the barrier method, allows to obtain second order differentiability of the value function and first order differentiability of the policy function, thus allowing for first order (log) linearization. Unfortunately, higher order perturbation methods (such as those in Uribe and Schmitt-Grohe, 2004) cannot be applied because there are not kwon results on higher order differentiability properties of the value function.

2.2 Approximation Methods

This paper uses two different types of polynomial approximation methods. The first one is piecewise linear interpolation (PLA). This interpolation method will be used only to achieve feasibility in the PFI algorithm for $K \subset \mathbb{R}^n, n \geq 2$. In particular, the papers uses a multidimensional simplical interpolation (see Judd, 1998, Pag. 243) to obtain any point in the endogenous state space. To achieve this result, a family of disjoint simplices is constructed. The second method will be used only for $K \subset \mathbb{R}$. It is a cubic variation diminishing spline approximation (VDSA) which allows to interpolate a strictly concave function in the C^2 class. This last method constitutes a difference with the quadratic shape preserving spline approximation presented by Schumaker (1983), which was used by Judd (1994) to develop a feasible version of the VFI. It turns out that a C^2 interpolant approximates a solution even in the boundary of the feasible region using the barrier method.

2.2.1 Simplical Multidimensional Linear Interpolation.

Let $K \subset R^n$ and $\{S^j\}$ be a disjoint finite family of simplices in K. Define the grid size, h, to be

$$h \stackrel{\text{def}}{=} \max_j diam\{S^j\}$$

Where $diam\{S^j\} \equiv \max_s\{\rho(s, s'); s, s' \in S^j\}$ and ρ to be the natural distance in the Euclidean space.

Defining K^j to be an arbitrary vertex of triangulation of the j^th simplex, any point $K \in K^j$ can be expressed as

$$K = \sum_j \lambda_j(K)K^j, \qquad \sum_j \lambda_j(K) = 1$$

The summation takes place over the vertices of S^j and $\lambda_j(K)$ is an scalar that is used to write any point in the state space, for instance K, as a convex combination of the vertices. Then, using Lagrange data on a function, $\{f^h(K^j); K^j\}$, PLA extends the observed points over the hole domain of the continuous endogenous state variable as follows,

4. $$f^h(K) = \sum_j \lambda_j(K) f^h(K^j)$$

Note that interpolation is only over K given the assumption of a discrete exogenous state space. Further, note that piecewise linear interpolation preserves the bounds of the interpolated function. That is, $f(x) \leq \bar{B} \rightarrow f^h(x) \leq \bar{B}$. This fact will be useful in the proof of first theorem.

2.2.2 Schoenberg's Variation Diminishing Spline Approximation

The importance of shape preservation and smoothness of the interpolating procedure should be clear from the perspective of the numerical optimization algorithm required to solve each step of the iterative process implied by the Bellman equation. For a general discussion on the subject the reader is referred to the operations research literature (Jhonson, 1993). Regarding dynamic macroeconomic models, as mentioned, Judd (1994) use the Schumaker's quadratic spline approximation to derive theoretical error bounds for a VFI algorithm. This analysis differs from Judd's in the preservation of the second order differentiability of the value function. This last feature allows convergence in the barrier method in a strong sense and the use of Gradient Hill Newton Methods. As in Judd (1994), the focus here is only on one endogenous state variable to allow for a shape preserving interpolating scheme.

In particular, VDSA enables to preserve the concavity and differentiability properties of the value function and Assumption 2 together with the barrier method will generate an optimization problem in the convex class.

The analysis on VDSA is based on a recent book by Lyche and Morken (2008).

Definition 1: Let $f: [a, b] \to \mathbb{R}$ be a continuous function, d a given positive integer and $t = [t_1, \dots, t_{n+d+1}]$ a nondecreasing sequence of knots with boundary knots $t_{d+1} = a, t_{n+1} = b$. The spline given by

5. $$(Vf)(x) = \sum_{j=1}^{n} f(t_j^*) B_{j,d}(x)$$

Here Vf is called Variation Diminishing Spline Approximation. Further, $t_j^* = \left(\sum_{i=1}^{d} t_{j+i}\right)/d$ and $B_{j,d}(x)$ is the j^{th} spline of order d defined by the usual recursive relation:

$$B_{j,d}(x) = \frac{x - t_j^*}{t_{j+d}^* - t_j^*} B_{j,d-1}(x) + \frac{t_{j+d+1}^* - x}{t_{j+d+1}^* - t_{j+1}^*} B_{j-1,d-1}(x)$$

Note that it is required at least d+2 knots to construct a spline of order d. So $Vf(x)$ is constructed with n polynomial pieces that connect each other, every one of them having d+2 knots. Finally, note that the condition on the end knots implies that the first and last d+1 knots are all equal. According to the recursive relation that defines the splines, this would imply division by zero, so it is usual to assume that "anything divided by zero is zero" (see Lyche and Morken, Pag. 30). This is done to allow all the pieces of the polynomial function to have d+2 knots.

This section turns now to the smoothness properties of the splines.

Theorem 2.2: Suppose that z occurs m times among the knots $t_j^*, t_{j+1}^*, \dots, t_{j+d+1}^*$ which define $B_{j,d}(x)$. If $1 \le m \le d+1$, then $D^r B_{j,d}$ is continuous at z for r=0,1,…,d-m but $D^{d-m+1} B_{j,d}$ is discontinuous at z.

Proof: see Lynche and Morken Theorem 3.19

So from Theorem 2.2 it is clear that to get a C^2 interpolant it is necessary to deal with a cubic VDSA and knots with multiplicity 1.

Finally this subsection establishes the monotonicity and shape preservation properties of the spline. The first property will be useful to preserve the monotonicity of a discrete Bellman operator, which, because of the Blackwell sufficient conditions, will be a contraction.

Theorem 2.3: if f is increasing on $[a, b]$, then Vf is also increasing in $[a, b]$.

Proof: see Proposition 5.32 and 5.33 in Lyche and Morken.

Theorem 2.4: if f is concave on $[a, b]$, then Vf is concave in $[a, b]$

Proof: see Proposition 5.37 and 5.38 in Lyche and Morken.

The intuition of Theorem 2.3 should be clear when recalling the value of the derivative the VDSA ($Vf'(x) = \sum_{i=1}^{n} d[f(t_i^*) - f(t_{i-1}^*)/t_{i+d}^* - t_i^*]$) and the fact that the sequence of knots is nondecreasing.

For Theorem 2.4 note that a concave increasing function will have decreasing derivatives, so in this case define,

$$\Delta c_i \stackrel{\text{def}}{=} [f(t_i^*) - f(t_{i-1}^*)/t_{i+d} - t_i] = [f(t_i^*) - f(t_{i-1}^*)/d(t_{i+d}^* - t_i^*)]$$

So if f is concave, $\{\Delta c_i\}$ should be a decreasing sequence. Theorem 5.37 in Lynche and Morken states that this condition is sufficient for Vf to be concave.

2.2 Barrier Methods

The combinatorial difficulty of inequality constraints problems induces to focus on the maximization problem that must be solved in each step of the Bellman operator defined in equation 2.

A recent taxonomy on numerical optimization algorithms (Nocedal and Wright, 2006, Ch. 15 to 17) considers several options to tackle the problem. However, the interest on the asymptotic properties of an algorithm requires a method with well-known convergence properties. Sometimes, inequality constrained problems are solved through a practical approach, meaning that these methods have no well defined convergence properties. Besides, the behavior of the Lagrange multiplier is clearly not trivial. In continuous state stochastic Markov Decision Problems, this means that the algorithm must keep track of the Lagrange multiplier over the hole discretized state space and during the number of iterations required to reach a desired level of the FPE. The numerical literature (Christiano and Fischer, 2000, Cao Alvira 2009) states that for a sufficiently small error tolerance, these facts can make the problem intractable. So given the need of an accurate convergent result, these problems must be avoided.

The literature has proposed several strategies to solve a constrained problem as a sequence of unconstrained problems.

The first is the Penalty Function Method (Courant, 1943). This method punishes the objective function every time the control variable leaves the feasible region. Although a convergence theorem for the equality constrained version had been proved, similar results for the inequality constraint case are not known. Furthermore, while it is usual to accommodate the penalty term through the square of the constraints (so the punishment is increasing in the violation), dealing with inequality constraints requires a max function. This affects the choice of the numerical optimization algorithm. In particular, it is not possible to use Gradient Hill optimization methods. Further, as the kinks in the objective function of the modified problem inherit the combinatorial difficulty of the problem, it is difficult to use algorithms for non-smooth optimization (Bazaraa, 1993, Ch. 8), which are suggested for piecewise functions with kinks in a non increasing number of points.

Another possibility is the use of Augmented Lagrangean Methods (Powell, 1969) or sequential quadratic programming (Nocedal, Ch. 18). However, these methods require an estimate of the Lagrange multipliers and, in models with state dependent constraints, the combinatorial difficulty of the problem will make them unfeasible.

This paper chooses to work with the Logarithmic Barrier Method. This method does not require the estimation of a Lagrange multiplier to find an approximate maximizer and have convergence results for inequality constrained problems. The main drawback of this method is its numerical performance. Interior Point Methods (Nocedal, Ch. 19) and Sequential Quadratic Programming are faster and more robust; however, both of them require an estimate of the Lagrange multipliers to solve the modified optimization problem.

Suppose the following inequality-constrained optimization problem[4],

[Problem 1] $\max_x f(x) \quad s.t. \quad c_i(x) \geq 0, \;\; i = 1, \dots, m$

Where $f: \mathbb{R}^n \to \mathbb{R}$, $c_i: \mathbb{R}^n \to \mathbb{R}$, f and c_i are at least continuous and the usual dimensionality restrictions have been applied to m and n. The set of functions that defines the inequality constraints form the feasible region, denoted as $\mathcal{F}$.

[4] The rest of this section relies heavily in Forsgren, Gill and Wright (FGW, 2002).

Instead of Problem 1, it is possible to solve a sequence of unconstrained problems, with a typical element,

[Problem 2] $$\max_x B(x, \mu_k) \stackrel{\text{def}}{=} f(x) + \mu_k \sum_{i=1}^{m} ln\big(c_i(x)\big)$$

where $B(x, \mu_k)$ is the logarithmic barrier function and μ_k is a positive scalar known as the barrier parameter. Note that $B(x, \mu_k)$ is as smooth as the objective and constraint functions.

Define then a sequence of maximizers, $\{x_k\}$, which solve the unconstrained problem as $\mu_k \to 0$. The next paragraph presents the conditions that guarantee $x_k \to x^*$, for any decreasing sequence $\{\mu_k\}$ as long as $\mu_k \to 0$, where x^* is the solution to Problem 1. Stronger assumptions also allow to define a differentiable path for $\{x_k\}$ and a convergent estimate of the Lagrange multiplier, which depends on the barrier parameter, the constraint functions and x_k.

Before quoting the results, consider an additional assumption and a definition.

Assumption 4: The feasible region, $\mathcal{F}$, is convex.

Sufficient conditions for assumption 4 are concavity or cuasiconcavity requirements on c_i. Assumption 4 allows to avoid some technicalities (see Definition 3.4 in FGW). However, if the feasible region is not convex, the barrier maximizers are still convergent.

Definition 2: Let N and N be sets in $\mathbb{R}^n$ such that N* ⊂ N. The subset N* is called isolated if there exists a closed set, E, such that N* ⊂ $int(E)$ and E⋂N=N*.*

Note that if N is compact and N=N*, then definition 2 is immediately satisfied (see FGW, Pag. 547).

Theorem 2.5: Let N be the set of maximizers of Problem 1, with objective function value f^*. Assume further that N is nonempty. Let $\{\mu_k\}_k$ be a strictly decreasing sequence of positive barrier parameters with $\mu_k \to 0$. Assume further that,

i) There exists a nonempty compact set N of local maximiers that is an isolated subset of N.*
ii) At least one point in N is in the clousure of $int(\mathcal{F})$*

Then,

a) There exists a compact set S such that N ⊂ $int(\mathcal{F})$ and such that for any $\bar{x} \notin \mathcal{F} \cap S$, $\overline{x} \notin N^*, f(\bar{x}) < f^*$*
b) For all $\mu_k < \varepsilon$, there exists y_k such that, $B(y_k, \mu_k) = max\{B(y_k, \mu_k) | x \in int(\mathcal{F}) \cap S\}$
c) For any $\{y_k\}_k$, $\exists \ \{y_{k,j}\}_{k,j} \to x_\infty$ with $x_\infty \in \ N^*$ and $lim_{j\to\infty} f(y_{k,j}) = f^* = \lim_{j\to\infty} B(y_k, \mu_k)$

Proof: see Fiacco and McCormick, 1990, Theorems 8 and 10.

Statement (a) is the formal definition of a constrained maximizer without requiring the maximal element to be a Khun Tucker point. Statement (b) defines a sequence of (bounded) barrier maximizers in the interior of a feasible region. Finally, statement (c) proves the existence of a convergent subsequence.

It must be noted that this theorem is too general for most economic applications (see Wright, 1992, for a weaker version of the theorem), where generally the existence of a Khun Tucker point and the compactness of the set of maximizers are generally assumed.

Theorem 2.5 allows the barrier maximizers to converge to the solution of Problem 1 requiring neither second order sufficient conditions nor constraint qualifications. This flexibility comes with a price: without sufficient conditions for isolated maximizers, it will be difficult, in practice, to keep track of the generated sequence. In particular, it is not necessarily true that every sequence of maximizers of the barrier function converges to the solution of Problem 1, in fact the theorem assures only the existence of a convergent subsequence.

All those considerations demand more restrictions to Problem 1.

Theorem 2.6: Let x^* be a local constrained maximizer of Problem 1, let $g(x^*), J(x^*), A(x^*)$ be the gradient of the objective function, the jacobian of the constraints and the set of active constraints at x^*. Assume further that,

i) x^* is a Kuhn Tucker point
ii) $J_A(x^*)$ has full row rank (Linear Independence Constraint Qualification, LICQ)
iii) Strict complementarity holds
iv) The second order sufficient conditions for an isolated constrained maximizer holds.

Then as $\mu_k \to 0$,

a) For any $\{y_k\}_k$, $\exists$ $\{y_{k,j}\}_{k,j} \to x^*$
b) Let $\mu_{k,j}/c_i(y_{k,j})$=λ_j^i denote the estimate for the Lagrange multiplier of constraint i. Then the sequence of barrier multipliers is bounded and $\lambda_j^i \to \lambda^*$, with $\lambda^{i,*} > 0$ if the i^{th} constraint is binding.
c) For a sufficiently large j, the barrier function is definite negative
d) A unique continuously differentiable function $x(\mu)$ of unconstrained barrier maximizers exists in a neighborhood of $\mu = 0$ and $x(\mu) \to x^*$.

Proof: see Theorem 3.12 and Lemma 3.13 of FGW.

Strict complementarity and LICQ are usual assumptions in the literature (see Santos and Rincon Zapatero, 2009). This allows to deal with a unique bounded λ^* and attain convergence for a sequence of locally unique maximizers of the barrier function (for sufficiently small values μ) that converge to an isolated solution of Problem 1, even for maximizers on the boundary of the feasible region. Further, it is also possible to approximate arbitrarily the Lagrange multiplier vector through a differentiable policy function. This last fact is important since it allows for the possibility of Perturbation Methods in the presence of binding inequality constraints.

Note the difference between Theorem 2.5 and 2.6; in the first one, the setting is so general that it is not possible to guarantee the existence of a Kuhn Tucker point or a finite set of Lagrange multipliers. As it is only assumed that $\mathcal{F}$ is convex, the usual (Mangasarian – Fromovitz or Linear Independence) constraint qualifications may not necessarily hold. Thus, in the context of Theorem 2.5 the Lagrangean function may not be well defined.

2.3 Fixed Point Functional Algorithm

This subsection introduces an unfeasible[5] version of the VFI and PFI, omitting details for simplicity. The reader is referred to Rust (1994) for a survey on numerical dynamic programming methods.

Algorithm 1 (VFI)

I) Select a tolerance level, ε, and an initial value function $W_0(k_0, z_0)$.
II) Compute the VFI,

$$W_{n+1}(k_0, z_0) = \max_{k' \epsilon \Gamma(k_0, z_0)} \left\{ u(k_0, k', z_0) + \beta \sum_{z' \varepsilon Z} W_n(k', z') Q(z', z_0) \right\}$$

III) End of iteration,

If $\|W_{n+1}(k, z) - W_n(k, z)\| < \varepsilon$, otherwise return to step II.

Making the algorithm feasible demands to derive an approximation of W, denoted as $W^{h(2)}$, where $h(2)$ denotes the VDSA. Define then a discretized Bellman operator ($T^{h(2)}$) associated with the discretized value function. This algorithm converges globally at a linear rate, β. Note that the VDSA is cubic, so the interpolated functions belong to the $C^2[a,b]$ class as long as the maximizers do not belong to the boundary of the feasible region. This fact, together with some concavity assumptions on the function that defines the restrictions, will suffice to apply Theorem 2.6. As it will be shown, $T^{h(2)}$ preserves the concavity of the value function, and together with the shape preserving property of the VDSA and part (c) of Theorem 2.6, these conditions assure that the maximization problem in the Bellman equation is in the convex class.

Algorithm 2 (PFI)

I) Select a tolerance level, ε, and an initial value function $W_0(k_0, z_0)$.
II) Policy Improvement step: Find $g_n(k, z)$ that solves

[5] Feasibility comes from the approximation methods introduce in section 2.2. As stated in this subsection, the algorithms assume the existence of an exact solution over the whole state space.

$$B(W_n)(k_0,z_0) = -W_n(k_0,z_0) + \underset{k'\epsilon\Gamma(k_0,z_0)}{Max} \left\{ u(k_0,k',z_0) + \beta \sum_{z'\varepsilon Z} W_n(k',z')Q(z',z_0) \right\}$$

Where B(.) is a functional operator that has a zero in the solution to the Bellman equation, W.

III) Policy Iteration step: find $W_{n+1}(k,z)$ *such that,*

$$W_{n+1}(k_0,z_0) = \underset{k'\epsilon\Gamma(k_0,z_0)}{Max} \left\{ u(k_0,k',z_0) + \beta \sum_{z'\varepsilon Z} W_{n+1}(k',z')Q(z',z_0) \right\}$$

IV) End of iteration,

If $\|W_{n+1}(k,z) - W_n(k,z)\| < \varepsilon$*, otherwise return to step II.*

Feasibility of the algorithm requires to derive an approximation of W, denoted as $W^{h(1)}$, where $h(1)$ denotes the PLA. As before, define a discretized Bellman operator ($T^{h(1)}$) associated with the discretized value function. This algorithm converges globally at a rate of 1.5 and locally at a quadratic rate. The quadratic result is inherited from the Newton-Kantorovich theorem, which is a root-finding method for functional spaces (see for instance Argyros, 2006). The local quadratic rate will not be used for several reasons: i) it requires strong assumptions, ii) the range of convergence has not been derived yet, iii) for a benchmark – first attempt algorithm, a global result is preferred. Theorem 2.5 and the Berge's Theorem of the Maximum will be used to solve the maximization problem in II) and to guarantee the existence of a policy correspondence.

3. Asymptotic Results for Numerical Dynamic Programming Algorithms with Inequality Constraints.

This section connects all the pieces presented in section 2, establishing the main results in two theorems. The first one, for $K \subset \mathbb{R}^n$, uses a results due to Santos and Rust (2004) who derived a computationally feasible version of the Puterman and

Brumelle (1979) theorem for the PFI Algorithm. The theorem uses a result due to Santos and Vigo Aguiar (1998) to set bounds on the approximation error. It then uses the barrier method (Theorem 2.5) to achieve an ε-approximation to the maximization problem. The second theorem derives a similar result $K \subset \mathbb{R}$ using the VFI. The theorem derives a feasible algorithm using VDSA and sets bounds to the approximation error using a theorem due to Cohen, et. al. (2001). Finally, theorem 2 achieves an ε-approximation of the maximization problem using Theorem 2.6.

It should be emphasized that there are worst case complexity results for continuous state infinite horizon MDP similar to those in Chow and Tsitsiklis (1991) that were provided by Rust (1994, see Theorem 5.2) and could be used to derive similar consistency results for the VFI algorithm. However, worst case complexity may serve as a stringent bound that can be relaxed using "the special structure of problems encountered in practice" (Rust, 1994, Pag. 153).

3.1 The Structure of the Feasible Algorithm

This subsection presents the algorithm exploiting the special structure of recursive dynamic macroeconomic models with inequality constraints in the Euler Class.

Given the interest in including inequality constraints in the optimization problem, Assumption 3 is no longer valid. However, note that the barrier method (Problem 2) has a solution in the interior of the feasible region. Further, Problem 2 is designed to handle only inequality constraints. This demands to assume that either the problem has no equality constraints or that it is possible to substitute them into the barrier and value functions.

Taking into account the inequality constraints, the Bellman equation can be written as,

6. $$W(k,z) = \max_{k'}\{u(k,k',z) + \sum_{i=1}^{m} \lambda_i(k,z)c_i(k,z) + \beta \sum_{z' \varepsilon Z} W(k',z')Q(z',z)\}$$

Where $c_i(k,z)$ denote the state dependent inequality constraints which, due to assumption 4, are not topologically inconsistent (see FGW, Definition 3.4). Further note that, for simplicity, the dependence of c_i on k' has been omitted.

Then rewrite equation 6 using the barrier method, just as in Problem 1,

7. $$W(k,z) = \max_{k'}\left\{u(k,k',z) + \mu(z) \sum_{i=1}^{m} ln\left(c_i(k,z)\right) + \beta \sum_{z' \varepsilon Z} W(k',z')Q(z',z)\right\}$$

Denoting $\mu(z)$ as the parameter associated with the initial state $z \in Z$, as it is possible to solve a different Bellman equation for each initial exogenous state. This is to stress that the level of the initial exogenous state variable plays a central role in determining whether the constraint is binding or not (see Christiano and Fischer 2000 for a related discussion).

Note also that the Lagrange multiplier function have been replaced with the barrier parameter, so now it is not necessary to estimate it to solve the problem. This constitutes a major difference with respect to most available numerical dynamic programming methods (like Marcet's PEA) or optimization algorithms (SQP and Interior Point) that deal efficiently with inequality constraints.

Despite this fact, as can be seen from Theorem 2.6, imposing some additional assumptions, allows to approximate arbitrarily not only the policy function but also the Lagrange multiplier function for k' in the clousure of $int(\mathcal{F})$. That is, if the maximizers in equation 6 lies on the boundary of $\mathcal{F}$ (i.e. there are at least one active constraint), it is possible to arbitrarily approximate the maximizer together with the associated multiplier as $\mu(z) \to 0$.

Finally, note that the constrained maximization problem in equation 6 have turned into a free one, with return function $u(.) + \mu(z)\sum_{i=1}^{m} ln\big(c_i(k,z)\big)$. Then, it is easy to see that the Bellman operator in equation 7, denoted as T, is also a contraction and have a unique fixed point (W).

The subsection turns now to the feasibility of the algorithm. Define the space of interpolating approximations, $\mathcal{W}^h$, as

$$\mathcal{W}^{h(1)} \overset{\text{def}}{=} \left\{W^{h(1)}: K \times Z \to \mathbb{R} \middle| W^{h(1)}\ is\ a\ piecewise\ linear\ approximation\ of\ W\ \in\ \mathcal{W}\right\}$$

$$\mathcal{W}^{h(2)} \overset{\text{def}}{=} \left\{W^{h(2)}: K \times Z \to \mathbb{R} \middle| W^{h(2)}\ is\ a\ variation\ diminishing\ \ approximation\ of\ W \in\ \mathcal{W}\right\}$$

where $W^{h(1)}$ and $W^{h(2)}$ are defined as in equations 4 and 5 respectively and the approximation of an element of $\mathcal{W}$ takes place only in the endogenous state variable, k.

Defined a discretized version of the Bellman Operator, $T^{h(i)}: \mathcal{W} \to \mathcal{W}^{\boldsymbol{h(i)}},\ i = 1{,}2$ as,

8. $$W_{n+1}^{h(i)}(k,z) = T^{h(i)}(W_n)(k,z) =$$
$$\max_{k'}\left\{u(k,k',z) + \mu(z)\sum_{i=1}^{m} ln\big(c_i(k,z)\big) + \beta\sum_{z' \varepsilon Z} W_n^{h(i)}(k',z')Q(z',z)\right\}$$

Using the Blackwell sufficient conditions, it is a standard exercise to show that $T^{h(i)}, i = 1,2$ is a contraction mapping with modulus β. So the VFI algorithm presented in section 2.3 immediately applies (see Algorithm 1) and converges at a liner rate, β. Further, the arguments in section 2.1 allow to show that there exists a unique fixed point of $T^{h(i)}, i = 1,2$, denoted as $W^{h(i)}$.

Moreover, define the operator $T_{\varepsilon}^{h(i)}: \mathcal{W} \rightarrow \mathcal{W}^{\boldsymbol{h}(\boldsymbol{i})},\ i = 1,2$, which satisfies,

$$\left\|T_{\varepsilon}^{h(i)}\left[W^{h(1)}(k,z)\right] - T^{h(i)}\left[W^{h(1)}(k,z)\right]\right\| \leq \varepsilon, \varepsilon > 0$$

with the usual Sup-Norm. Here $T_{\varepsilon}^{h(i)}$ is used to set bounds to the numerical errors of the barrier method in the maximization problem implicit in the Bellman equation 8. This bound guarantees an ε-approximation of $W^{h(i)}$. As before, $T_{\varepsilon}^{h(i)}\ i = 1,2$ is a contraction mapping with modulus β (see Santos, 1999).

Now, from Santos (1999), the following inequality shows a decomposition of the error involved in a general numerical dynamic programming algorithm,

9. $\left\|W - T_{\varepsilon}^{h(i)}\left[W_n^{h(i)}\right]\right\| \leq \left\|W - W^{h(i)}\right\| + \left\|W^{h(i)} - T^{h(i)}\left[W_n^{h(i)}\right]\right\| + \left\|T^{h(i)}\left[W_n^{h(i)}\right] - T_{\varepsilon}^{h(i)}\left[W_n^{h(i)}\right]\right\|$

where W and $W^{h(i)}$ are fixed points of T (equation 7) and $T^{h(i)}$ (equation 8) respectively.

Intuitively, it is possible to achieve an arbitrarily approximation of W by deriving consistency theorems for the FPE (the second term in the right hand side of the inequality) and the ME (the third term). This terminology is due to Rust (1994). The first term is the APE, which will be bound using standard arguments in the polynomial approximation literature.

3.2 Main Results

This subsection derives a general theorem for $K \subset \mathbb{R}^n, n \geq 2$, using the PFI algorithm and Theorem 2.5 to tackle the FPE and ME respectively. Theorem 4.3 in Santos (1999) deals with the APE. The theorem will allow to work with inequality constraints and an arbitrary number of endogenous state variables, achieving convergence for the FPE at a rate of 1.5. Then the paper turns to a problem with only one endogenous state using the VFI algorithm, Theorem 2.6 and VDSA to set bounds to the FPE, ME and APE respectively.

Theorem 3.1: Let $W^{h(1)}$ be the fixed point of $T^{h(1)}$ and W be the fixed point of T. Assume that $W^{h(1)}$ is concave. Let $\left\{W_n^{h(1)}\right\}_{n\geq 1}$ be the sequence of value functions generated by the PFI (Algorithm 2) over a uniform grid. Then under Assumptions 1 and 2:

$$\left\|W - T_\varepsilon^{h(1)}\left(W_{n+1}^{h(1)}\right)\right\| \leq \frac{\gamma h^2}{2(1-\beta)} + \left[\left(\frac{\beta^{1.5}}{1-\beta}\right)\left(\frac{4}{h}\right)\left(\frac{1}{\eta^{0.5}}\right)\right]^{n+1} M^{1.5} + \varepsilon$$

where γ is the bound of $D^2 W$ in Theorem 2.1, h is the grid size and η is the parameter that defines the α-concavity of the return function, M is the bound of the value function in equation 1 and $T_\varepsilon^{h(1)}$ is the Bellman operator that solves the maximization problem through the barrier method (Theorem 2.5) and thus has a bounded numerical error, ε.

Proof: see the Appendix

The first term is the APE for a PLA (which decreases in h), the second is the FPE for the PFI algorithm (which increases in h) and the last term is the ME. Finally Theorem 2.5 allows to approximate arbitrarily the policy function in equation 6.

Theorem 3.1 represents an extension of the results in Santos and Rust (2004), as it allows to derive theoretical error bounds for recursive dynamic models without imposing interiority assumptions. Besides, in contrast with previous results (Santos and Rust, 2004, Judd, 1994, Marcet and Marshall, 1994), the bound presents the dependence of the three sources of error on the deep parameters of the model, thus allowing to obtain the desired level of accuracy together with a convergence rate.

Theorem 3.2: Let $W^{h(2)}$ be the fixed point of $T^{h(2)}$, W be the fixed point of T and $K \subset \mathbb{R}$. Let $\left\{W_n^{h(2)}\right\}_{n\geq 1}$ be the sequence of value functions generated by the VFI (Algorithm 1) over a uniform grid. Assume that the conditions ii and iii of Theorem 2.6 are satisfied. Then under standard Assumptions 1 and 2:

$$\left\|W - T_\varepsilon^{h(2)}\left(W_{n+1}^{h(2)}\right)\right\| \leq \frac{h^2}{(1-\beta)} + \left[\left(\frac{\beta}{1-\beta}\right)\right]^{n+1} M + \varepsilon$$

$$lim_{\mu(z)\to 0}\, \lambda_{K,Z}^{i} = \lambda_{K,Z}^{i,*} \text{ with } \lambda_{K,Z}^{i,*} > 0 \text{ if the } i^{th} \text{ constraint is binding.}$$

where h is the grid size, M is the bound of the value function and $T_{\varepsilon}^{h(2)}$ is the Bellman operator that solves the maximization problem through the barrier method (Theorem 2.6).

Proof: see Appendix

The first term is the APE for a VDSA (which decreases in h), the second is the FPE for the VFI algorithm (which does not depend on h) and the last term is the ME. Finally, note that Theorem 2.6 allows to approximate arbitrarily the policy function and Lagrange multiplier[6] in equation 6.

Theorem 3.2 constitutes an extension of the results in Judd (1994) as it does not require an interiority assumption. Besides, assures the existence of a differentiable convergent path for the barrier maximizers and an estimate of the Lagrange multipliers even for problem with multiple binding constraints. This last result constitutes a major difference with the existent literature, as smooth approximation methods (which estimate Lagrange multipliers as well as maximizers) do not have error bounds and discrete approximation methods (which have error bounds) do not estimate Lagrange multipliers (the interiority assumption makes them trivially zero). The importance of an estimate of the Lagrange Multiplier function in practice can be seen in Cao Alvira (2009), which finds a better fit of the velocity of money to data in a Cash-in Advance model allowing for occasionally binding constraints. This paper can provide a benchmark solution to the Cao Alvira's model (which uses a smooth approximation method), as it provides a globally convergent arbitrarily accurate solution.

Remark 1: Note that Theorem 3.1 and Theorem 3.2 implies that $\|g-\hat{g}\| \leq \delta, \delta > 0$, where g and $\hat{g}$ are the policy functions that solve equation 6[7] and 7 respectively. Then under Lemma 3.1 in Santos (2000) implies,

$$\left\|W - W_{\hat{g}}\right\| \leq \frac{H\delta\bar{\varepsilon}}{(1-\beta)}$$

where $\bar{\varepsilon}$ and H are constants defined in equation 3.4 and Assumption 4 of the mentioned paper.

[6] Note that the estimate of the Lagrange multiplier and the barrier parameter are both functions of the initial endogenous state. So when we write $\mu(z)$, we are doing an abuse of notation. This is done to stress the importance of the exogenous state in determining whether the constraint is binding.
[7] If Lagrange multipliers are not well defined we must use equation 1.

Because the (exact) Bellman operator in equation 7 is a contraction, by the (intractable) VFI algorithm of section 2.1 $W_{\hat{g}}$ is concave. Then, Remark 1 holds only from first order differentiability of the value function $W_{\hat{g}}$ and a bound on the composite mapping $\hat{g}\big(\hat{g}(\dots(\hat{g}(k_0, z_0)\dots)\big)$, which define the bounds for $\bar{\varepsilon}$ and H respectively. The first assumption is assured because the barrier maximizers belong to $int(\mathcal{F})$ and Assumption 2. The second is guaranteed by the Maximum Theorem, which is used in the proofs of Theorems 3.1 and 3.2 and assures the compactness of the policy function that solves equation 7.

Remark 1 states an error bound between the exact unfeasible Bellman operator (equation 6) and their modified version using the barrier method (equation 7). The result highlights the importance of the interiority of the barrier maximizers, as it allows to obtain a bound for the modified optimization program using standard approximation theory.

Remark 2: Note that Theorem 3.2 implies that for $\mu \to 0$, $\widehat{g_n}$ *is differentiable and the Hessian of the objective function in the maximization problem in equation 8,* $u(k, k', z) + \mu(z) \sum_{i=1}^{m} ln\big(c_i(k, z)\big) + \beta \sum_{z' \varepsilon Z} W_n^{h(2)}(k', z') Q(z', z)$, *is negative definite.*

Remark 2 states that, for small values of the barrier parameter, the modified optimization problem in equation 8 is in the convex class, thus, it can be solved in polynomial time. Remark 2 is an extension of the results in Judd (1994) for models with inequality constraints. In Judd's paper the concavity of interpolated value function was guaranteed by a quadratic (shape preserving) spline approximation, here VSDA is used to serve that purpose. Moreover, Remark 2 states that the policy function is differentiable only up to the first order. This finding allows for the possibility of (local) first order perturbation methods, such as those used by DYNARE (one of the most popular software packages used to solve DSGE models). This result differs sharply with the ones in Judd (1994), as the quadratic spline approximation used there does neither assure the second order differentiability of the value function nor, by Theorem 3.2 in Gallego (1993), the first order differentiability of the policy function. This remark can be used to contrast the results in Roca and Preston (2007), which uses the barrier method and second order perturbation algorithms in a heterogeneous agent RBC model with inequality constraints.

4. Conclusions.

Theorem 3.1 shows the existence of error bounds, which allow to approximate the fixed point in equation 7, where the constrained maximization problem in the exact Bellman operator (equation 1) have been modified by the barrier method. The distinct feature of equation 7 is the interiority of the approximate maximizers. Note that this last fact is crucial to set bounds to the APE, as most existing theory deals only with smooth functions.

Another remarkable aspect of Theorem 3.1 is that only requires very weak assumptions, which are completely standard for the literature. This fact is inherited from Theorem 2.5. Note that an application of the Maximum Theorem is enough to guarantee the existence of a solution in equation 7 and the convexity of the feasible region together with the compactness of the policy correspondence are sufficient for the existence of a convergent subsequence of barrier maximizers. The only restrictive assumption is the concavity of the fixed point in equation 8, which is required by the bound to the FPE. In fact, in practice it would be quite difficult to assure the existence of such a function because; i) only a finite number of steps in the PFI algorithm are possible, ii) multidimensional interpolation is not shape preserving. So, in practice, the degree of concavity in the approximate fixed point of equation 8 must be checked by some index, as the one developed in Johnson (1993).

Other distinctive fact of Theorem 3.1 is the convergence rate in the FPE, which is derived from the global result in Santos and Rust (2004). Even though Theorem 5.8 in Rust and Santos (2004) allows to achieve convergence at a local quadratic rate, the lack of existing theory on the domain of attraction (see Argyros 2008, Ch. 2 and 3) makes the algorithm unattractive. This is because the available theory deals with a Newton Kantorovich iterative algorithms. In the case of the PFI, the Frechet differentiability of the Bellman operator is not guaranteed during the iterative procedure. This fact requires a change in the Newton Kantorovich iterations, replacing the derivative with the support of the Bellman operator (see Santos and Rust, 2004 Pag. 2102). Thus, without theory on the domain of attraction, results may be divergent, especially because the topology of the domain may be not well defined (see for instance Polyak, 2004).

Moreover, note that the FPE is increasing in the grid size. The result is inherited from Theorem 5.7 in Santos and Rust (2004). This may explain why PFI works better with a thicker grid, comparing its performance with the VFI (see Rust, 1994, Ch. 5 section 4).

Finally, note that the APE in Theorem 3.1 depends on (the bound of) the Hessian of the value function. As theorem 3.1 deals with (first-order) linear approximations, only the first derivative is well defined. The simplicity and efficiency of the approximation algorithm affects its accuracy, as the error is increasing in the curvature of the function.

Theorem 3.2 imposes additional assumptions (*ii and iii* in theorem 2.6 and the restriction on the size of the state space) to guarantee the existence of a unique Lagrange multiplier and isolated (locally unique) maximizers of the constrained problem. This strengthen the convergence properties of the barrier method (Theorem 2.6), allowing to derive an asymptotic theory for the estimates of the Lagrange multipliers, a maximization problem in the convex class and a path of differentiable unique barrier maximizers (see Remark 2).

From a practical point of view, these last findings are significant. First, the maximization problem in equation 8 can be solved in polynomial time, meaning that the number of operations required to solve the optimization problem is bounded by a polynomial of finite order. For non-convex problems, numerical optimization algorithms are affected by the course of dimensionality. Consequently, the required number of operations is bounded by a constant that grows exponentially with the size of the grid. Second, there is no need to worry for the existence of multiple paths in the barrier problem. Third, this approximation allows to deal with occasionally binding constraints easily, as Lagrange multipliers can be estimated as a function of the initial state. This constitutes a major difference from Theorem 3.1.

On the other hand, Theorem 3.2 only allows for linear convergence of the FPE. This is because faster and more accurate algorithms (i.e. the PFI) do not preserve the concavity of the value function. The shape preservation is a crucial fact in the proof of Theorem 3.2, because it allows to apply Theorem 2.6, especially to prove assumption iv.

Further, note that the APE is smaller in Theorem 3.2 than in Theorem 3.1,given the use of a cubic smooth interpolant rather than a piecewise linear approximation.

Remark 1 establishes an error bound for the (exact) barrier Bellman equation. This is, only asymptotic approximation of the value function in equation 6 is possible through the fixed point in equation 7. Again, the interiority of the policy function in equation 7 plays a crucial role here, because it allows the differentiability of the value function, which is required to measure the error involved in the approximation.

Finally, regarding the numerical properties of the algorithm, as is well known from the literature (see for instance Rust, 1994, Ch. 5) VFI and PFI are among the most accurate algorithms available. However, they are two of the most inefficient ones as measured in CPU time. Further, the numerical properties of the logarithmic barrier method are not quite satisfactory either. This is because the method is very ill conditioned when it is solved through Quasi Newton and Direction Set methods and it also suffers from the Scaling Problem (Nocedal, Ch. 12) when solved through the Newton Method. This last fact can be explained by the barrier parameter. Because convergence demands the barrier parameter to be close to zero, it affects the size of the inverse of the Hessian of the objective function, and thus the numerical performance of the method. All these features increase the size of the ME, measured by ε in inequality 9. So, even though the size of the ME is bounded, it can be quite large, affecting the applicability of the algorithm. To measure the size of ε, a numerical test must be performed on the algorithm.

To conclude, as the objective of the paper is to derive theoretical error bounds for a global accurate algorithm, efficiency and numerical issues are not being taking into account seriously. In practice, the algorithms developed here can be used to complement more efficient local methods such as those used by Christiano and Fischer (2000) by providing an arbitrarily accurate benchmark solution. Moreover, Theorem 1 provides an extension of the Santos and Rust (2004) results, as it derives error bounds for an arbitrarily number of states and constraints without an interiority assumption. Further, Theorem 2 is an extension of the results in Judd (1994), as it derives a unique differentiable set of barrier maximizers that converge to the solution of the constrained problem. This last result can be applied to derive first order perturbation (local) methods.

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Appendix

Proof of Theorem 3.1

The Proof proceeds from left to right in inequality 9.

Part 1: Bound of the Maximization Error

Note that the following problem is being solved,

A.1 $W^{h(1)}(k,z) = \max_{k'\in\Gamma[k,z]}\left\{u(k,k',z) + \beta \sum_{z'\varepsilon Z} W^{h(1)}(k',z')Q(z',z)\right\}$

where k' is assumed to belong to the compact set that defines the state space, S. Further, it is known from Assumption 1 that the feasibility correspondence is continuous and convex valued. To guarantee the existence of a solution and the continuity of the value function, it is possible to use Berge's Theorem of the Maximum. So the continuity of the sequence of value functions generated by algorithm 1, $\left\{W_n^{h(1)}\right\}_{n\geq 1}$ must be checked.

Note that step III in Algorithm 2 solves for $W_{n+1}^{h(1)}$. To accomplish this feature in a finite state space, the algorithm solves the following system of equations,

A.2 $$W_{n+1}^{h(1)} = u\left(k, g_n^{h(1)}, z\right) + \beta P_{g_n^{h(1)}} W_{n+1}^{h(1)}$$

Where $P_{g_n^{h(1)}}$ is the Markov operator in the problem, which can be written in matrix form. To check this last statement see Lyche y Morken (2008), Ch. 2 Section 3 for the algebraic manipulations required for PLA and note that it is always possible to write the transition probabilities in a finite state first order Markov process as a matrix.

Equation A.2 can be written equivalently as,

A.3 $$\left[I - \beta P_{g_n^{h(1)}}\right] W_{n+1}^{h(1)} = u\left(k, g_n^{h(1)}, z\right)$$

So to solve A.3, it is necessary to invert the matrix $\left[I - \beta P_{g_n^{h(1)}}\right] \stackrel{\text{def}}{=} A$.

In order to do that, the full row rank of A must be checked. The Banach inversion lemma on lineal operators (see Argyros, Pag. 4) can serve this purpose.

Assume that the maximization problem in A.1 has a well defined solution for $n = 0$. This is typically assured with the proper choice of $W_0^{h(1)}, k_0, z_0$. Note that $P_{g_n^{h(1)}}: \mathcal{W} \to \mathcal{W}^{h(1)}$ is a linear, positive, bounded operator.

Let us define $Q \stackrel{\text{def}}{=} I$, the identity matrix and note that $[I - QA] = \beta P_{g_n^{h(1)}}$.

To show that A[-1] exists, following Banach's inversion lemma, it must be shown that $\|I - QA\| < 1$, where the Sup-Norm is being used, as in Santos and Rust (2004, Pag. 2102, Equation 4.9). Let a denote a typical scalar in A. Then, because A is formed by the weights in a PLA and the elements of a transition matrix, $a \in [0,1]$. So, $\beta \in (0,1)$ is sufficient to establish the desired result.

Finally note that the Banach inversion lemma together with equation A.3 implies,

A.4 $$W_1^{h(1)} = A^{-1}\left[u\left(k, g_0^{h(1)}, z\right)\right]$$

As $u\left(k, g_0^{h(1)}, z\right)$ is a continuous function and A^{-1} is a linear operator, the continuity of $W_1^{h(1)}$ has been established.

Having proved the continuity of $W_1^{h(1)}$, to apply Berge's Theorem of the maximum, the continuity and compactness of the feasibility correspondence must be verified. This is done with Assumptions 1 and 2.

So, the continuity of the sequence of value functions generated by Algorithm 2 and the upper hemi continuity of the associated policy functions, $\left\{W_n^{h(1)}\right\}_{n\geq 1}$ and $\left\{g_n^{h(1)}\right\}_{n\geq 1}$ respectively have been proved. Note also that $g_n^{h(1)}, n \geq 1$, is also compact valued.

Note that the same arguments that were used to establish the continuity and compactness of the value and policy functions in A.1, can be used in equation 8. So, once the convergence of the barrier method has been proved, the continuity and compactness proof relevant for this equation will be omitted.

The paper turns now to the conditions in Theorem 2.5, which are required to solve the maximization problem in equation 8.

Setting $g_n^{h(1)} = N$ and choosing $N = N^*$ for the sets defined in Theorem 2.5, it is clear that N^* is an isolated subset of N. So assumption i is satisfied.

Further as, $g_n^{h(1)}$ is compact valued, feasible and $N = N^*$ it is clear that at least one point in N^* is in the clousure of $int(\mathcal{F})$, so assumption ii is also satisfied.

So part c) of Theorem 2.5 assures the existence of a convergent subsequence of barrier maximizers with a limit point in N^*. This limit point will be denoted $\hat{g}_n^{h(1)}$.

So it has been established that the barrier maximizers can approximate arbitrarily the solution to the feasible constrained problem, equation A.1. Consequently, it is clear that the ME depends only on numerical (rounding, etc.) errors. Finally, note that $T_\varepsilon^{h(1)}$ is also a contraction mapping (see Santos, 1999, Pag. 328), so the sequence of ME converges to $\frac{\varepsilon}{(1-\beta)}$.

■

<u>Part 2: Bound of the Fixed Point Error</u>

The paper turns now to the bound affecting the FPE, using Theorem 5.7 in Santos and Rust (2004). Assumptions 1 and 2 (Pag. 2099) must be verified in equation 8.

Note that, assuming that it is possible to get rid of the boundary restrictions through Inada like conditions, the Bellman equation in 8 implies an unconstrained maximization problem with return function $u(k, k', z) + \mu(z) \sum_{i=1}^{m} ln\big(c_i(k, z)\big) \stackrel{\text{def}}{=} l(k, z)$. So under Assumptions 1 and 2 in section 2 it is easy to see that the state space is compact, the feasibility correspondence is convex and compact valued and $l(k, z)$ is strictly concave for sufficiently small values of $\mu(z)$. So using Theorem 5.7 in Santos and Rust (2004) the sequence of value function under the PFI in equation 8, $\left\{W_{n,\hat{g}}^{h(1)}\right\}_{n\geq 1}$, satisfies,

A.5 $$\left\|W^{h(1)} - W_{n+1,\hat{g}}^{h(1)}\right\| \leq \frac{\beta L}{(1-\beta)} \left\|W^{h(1)} - W_{n,\hat{g}}^{h(1)}\right\|^{1.5}$$

Where $W^{h(1)}$ is the fixed point in equation 8, which was assumed concave.

After a simple iterative procedure and assuming $W_{0,\hat{g}}^{h(1)} = 0$ A.5 turns into

$$\left\|W^{h(1)} - W_{n+1,\hat{g}}^{h(1)}\right\| \leq \left[\frac{\beta L}{(1-\beta)}\right]^{n+1} \left\|W^{h(1)}\right\|^{1.5}$$

As the Sup-Norm is being used and PLA preserves the bounds of the value function in equation 7,

$$\left\|W^{h(1)}\right\| \leq \left\|SUP_{z\in Z}SUP_{k\in[K_L,K_U]}SUP_{k'\in\Gamma}\left[\frac{u(k,k',z)}{(1-\beta)}\right]\right\| \stackrel{\text{def}}{=} M$$

Note that the existence of M is assured because $T^{h(1)}$ is defined over the space of bounded continuous functions.

Finally, the constant in inequality A.5 is defined for a uniform grid by $L = 4\beta^{0.5}/h\eta^{0.5}$ (see Santos y Rust, 2004, Pag. 2105). So after a few algebraic manipulations,

$$\left\|W^{h(1)} - W^{h(1)}_{n+1,\hat{g}}\right\| \leq \left[\left(\frac{\beta^{1.5}}{1-\beta}\right)\left(\frac{4}{h}\right)\left(\frac{1}{\eta^{0.5}}\right)\right]^{n+1} M^{1.5}$$

■

Part 3: Bound of the Approximation Error

To establish a bound in the APE Lemma 3.4 in Santos and Vigo Aguiar (1998) is used.

Assumptions 1 and 2 of the mentioned lemma (Pag. 411) have already been verified. Assumption 4 is not necessary. To verify Assumption 3, the proof uses Lemma 3.7 in FGW (2002, Pag. 547), which assures the interiority of the solutions to equation 7. Then using Theorem 2.1 in section 2, D^2W is bounded, where W is the fixed point of the implicit Bellman operator in equation 7. Let γ denote that bound. So a simple application of Lemma 3.4 in Santos and Vigo Aguiar (1998) implies that,

$$\left\|W - W^{h(i)}\right\| \leq \frac{\gamma h^2}{2(1-\beta)}$$

■

■■

Proof of Theorem 3.2

Only the first and third part of inequality 9 in Theorem 3.2 will be proved as the second part is a simply application of equation 3, which is standard in the literature.

Part 1: Bound of the Maximization Error

Using Theorem 2.6 it will be shown that the sequence of barrier maximizers in equation 8 for $K \subset \mathbb{R}$ approximate arbitrarily the solution to equation A.1.

Suppose, as before, that $W_0^{h(2)}, k_0, z_0$ was chosen so as to allow the maximization problem in A.1 to have a solution. An application of the Theorem of the maximum is sufficient to establish the continuity of $W_1^{h(2)}$ and the upper hemi continuity of $g_0^{h(2)}$. Further, because under the VFI algorithm, $T^{h(2)}$ maps the space of bounded continuous strictly concave into itself, it is possible to guarantee that $g_0^{h(2)}$ is a continuous function.

Let us denote $\left\{W_n^{h(2)}(k^i, z), k^i\right\}_{i=1}$ the sequence of data points use to generate $VfW_n(k, z)$ in Definition 1 of section 2. Then $W_n^{h(2)}(k, z) = VfW_n(k, z)$ is increasing, strictly concave and belong to the C[2]-Class because of Theorems 2.3, 2.4 and 2.2 in section 2 respectively.

Remember that a sufficient condition for Assumption 4 is that the set of constraints functions, $c_i, i =, \dots, m$, are twice continuous differentiable and strictly concave. So under that assumption, the Hessian of the Lagrangean that can be defined from equation A.1 is negative definite, satisfying Assumption *iv* of Theorem 2.6.

Assumption *ii and iii* hold by hypothesis and *i* is guaranteed by the LICQ as the Lagrange multiplier, λ^i, is finite and unique for $i = 1, \dots, m$.

Finally, the convergence of the estimate of the Lagrange multiplier, $\lambda^i_{K,Z}$, and the negative definiteness of the barrier function in equation 8 are a direct consequences of part b and c in Theorem 2.6 respectively.

■

Part 3: Bound of the Approximation Error

In order to establish a bound in the APE, a standard result in the literature of spline approximations will be used (see Ch. 9 in Cohen, et. al. 2001).

The error in approximating a function in the C^2-Class with a cubic VSDA is $O(|\{t_i\}|^2)$, where $|\{t_i\}|^2 = max\{t_{i+1} - t_i\}$. Let t_i be an arbitrary average knot in Definition 1 of section 2, the uniformity of the grid implies $|\{t_i\}|^2 = h^2$.

Let W and $W^{h(2)}$ be the fixed point of T and $T^{h(2)}$ in equation 7 and 8 respectively. Then,

$$\left\|W - W^{h(2)}\right\| = \left\|TW - T^{h(2)}W^{h(2)}\right\| \leq \left\|TW - T^{h(2)}W\right\| + \left\|T^{h(2)}W - T^{h(2)}W^{h(2)}\right\|$$

Further,

A.6 $$\left\|W - W^{h(2)}\right\| \leq \left\|TW - T^{h(2)}W\right\| + \beta\left\|W - W^{h(2)}\right\|$$

Finally, inequality A.6 together with $|\{t_i\}|^2 = h^2$ implies

$$\left\|W - W^{h(2)}\right\| \leq \frac{h^2}{(1-\beta)}$$

■

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