Endogenous Switching of Volatility Regimes: Rational Expectations and Behavioral Sunspots

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Abstract

The importance of central bank communication policies and statistical learning in expectations formation have been recently emphasized. The present work merges and innovates basic ideas from both approaches in two respects. Firstly, we analyse a Lucas-type monetary model where private sector expectations are influenced by two, and not only one, institutional forecasters. Strategic motives takes place because the rational expectations equilibrium (REE) arises as solution of the simultaneous coordination game played by such big actors. Therefore, and this is a second novelty, both institutional forecasters have to learn not only about fundamentals but also about the rationality of the other's expectations. We show that the use of constant gain learning algorithms by institutional forecasters can give rise to endogenous, unpredictable and persistent switches in volatility regimes. Specifically, inflation dynamics can suddenly switches from the unique REE to a behavioral sunspot equilibrium and viceversa.

Keywords: expectations co-movements, excess volatility, adaptive learning, bounded rationality.

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1 Introduction

1.1 Changes in volatility regimes

This paper aims to provide a stylized model on how unpredictable and endogenous changes of volatility regimes can arise mainly because agents fail to form expectations independently. Volatility is one among the most important sources of uncertainty. Generally the higher and more frequent are fluctuations in the economy the higher are costs paid in terms of insurance or financial fragility. One of the most challenging task for economists is to understand when and how an high volatility crisis triggers.

The issue of excess volatility has recently received attention by the profession with special regard for US time series evolution after the second word war onwards (figure below).

Figure 1: US inflation (percentage change of Production Prices Index) time series of last fifty years. Grey bars denote recession periods. The picture suggests different volatility regimes with no strict correlation between fluctuation amplitude and growth cycles.

Cogley and Sargent (2005a), Sims and Zha (2006), Primiceri (2005) have found several different drifting and volatility regimes. These studies seems to give little importance to quantitative effect of monetary policy. Sims and Zha (2006) notice that:

"...the work of Cogley and Sargent and Primiceri all fits with the notion that the data do not deliver clear evidence of parameter change unless one imposes strong, and potentially controversial, overidentifying assumptions.".

Paying such price, Cogley and Sargent (2005b) and Primiceri (2006) gives an explanation volatility changes in terms of learning by the central bank. In their study there is evidence that high inflation in 70's would had risen because some years passed for the FED to correctly identify the model. In other words, Lucas' lesson would have been learned after evidence of it has been produced by the implementation of wrong policies. Nevertheless, even if partially in conflict, all these findings are in line with the suggestion coming from the picture above that high volatility periods and recessions are not significantly correlated. This would address the issue of volatility inflation changes to explanations not so strictly linked to cyclical real economy determinants.

1.2 Behavioral uncertainty, interdependent expectations and learning

An important part of the profession places now more and more emphasis on the role of central bank as focal point for agents' expectations, stressing the pre-eminence of the communication policy on the mere control of monetary determinants of the economy (a good introduction to the issue is Morris and Shin (2007)). Agents look at central bank expectations because everyone knows all others are looking at it, so that, central bank expectations provides noisy information on what the others are simultaneously expecting. This simple empirical fact tells economic theory that the idea individuals are able to hold rational expectations *independently* cannot hold. If this was the case, each one would simply have all relevant information and signals coming from the central bank would be just redundant.

Adaptive learning (Marcet and Sargent 1989, Evans and Honkapohja 2001) answers the need to design a more reasonable and dynamic theory of expectation formation in contrast with dogmatic acceptance of rational expectations hypothesis. The central idea is that agents act as econometricians. They form expectations according to a theory (a perceived law of motion) that is calibrated estimating recursively the impact of exogenous variables as data become available in real time. This more realistic way to think about expectations as a further dimension of the dynamics of the system introduces new issues as the use of misspecificated theories (Evans, Honkapohja and Sargent 1989, Sargent 1999, Evans and Honkapohja 2001) and evolutionary competition among alternative statistical predictors (Branch and Evans 2006, 2007, Guse 2005).

In this paper, both basic ideas, namely the one about the importance of the coordination role of central bank and the one about adaptive formation of expectations, are merged[1] and innovated in two respects to explain changes in volatility regimes. We analyse the setting in which, firstly, more than one, in this case two, institutional forecasters polarize private sector expectations and, secondly, professional forecasters use adaptive learning not only to learn about fundamentals but also to assess rationality of the other institutional forecaster's expectations.

Rating agencies, market leaders, fiscal authorities generally influence private sector expectations as well, and sometimes more, than the central bank. In general, whenever more than one agent has non-negligible impact on the aggregate expectation, holding rational expectations is a best action if and only if all others do the same. Therefore, because behavioral uncertainty, agents have the incentive to understand how others' expectations affect the actual economic course. The interaction entailed by the coordination expectation game among institutional forecasters is a first source of expectations interdependence. A second one is entailed by the role of institutional forecasters acting as focal point for private sector's expectations. The latter is a one-way dependence linking private sector's beliefs to institutional forecasters' expectations whereas the former is a

[1]Both arguments could have a further point of contact in the idea that estimation is a costly activity. The most part of agents cannot solve individually their own forecasting problem because the great amount of resources (at least cognitive) needed in gathering and processing all information to "produce" statistically consistent prediction. Therefore, agents look at central bank that maintains sufficient resources to form expectations according, in the best of cases, to an optimal statistical analysis of available data in light of the right theory underlying the working of the economy.

reciprocal interdependence among institutional forecasters' expectations.

In this paper, particular emphasis is placed on interaction between constant gain adaptive learning and these two forms of expectations interdependence being among primary reasons for persistent excess volatility triggering. To the aim we will simplify the setting in order to enlighten the basic mechanism and to make the analysis rigorous but handy. We don't want to neglect other determinants of excess volatility, but we aim to covey a first idea on how behavioral uncertainty alone can be enough to generate endogenous and unpredictable switches in volatility regimes.

1.3 Learning and communication

This work is a natural extension of Gaballo (2009). That paper investigates the learning dynamics of two institutional forecasters affected by behavioral uncertainty, but perfectly informed about both the exogenous determinants and the self-referential nature of the economy. Behavioral uncertainty arises in the sense they only have noisy observations of the simultaneous expectation of the other agent. Each agent estimates a coefficient weighting the noisy signal in expectation formation process responding to her own incentive to refine their forecasts. An equilibrium requires expectations to be locally optimal linear projections given behavioral uncertainty restrictions on the information set. Rational expectations equilibrium occurs whenever the estimated coefficient is zero, so that the behavioral noisy information is discarded. Otherwise, because the endogenous and interactive working of the learning algorithm, some equilibria different from REE can arise entailing excess volatility regimes. Those equilibria have been tagged behavioral sunspots equilibria (BSE). They configure as a coordination failure in that both agents use irrelevant information.

In the present study, this scheme governs the arising of endogenous interdependence among institutional forecasters' expectations and it is implemented in a simple monetary Lucas-type model (the same of Evans and Branch (2006)). The microfoundation of the model is presented in the first section, nevertheless the results are not linked to the particular specification at hand. There are no novelties in the model in se. The setting is very simple but it obeys to general economic incentives and constraints such as transversality conditions and non negativity of prices. This is enough to deal with one among the primary concerns of this paper, that is, to defend, in principle, economic relevance of BSE.

Four new directions are explored. First, institutional forecasters learn not only about others' rationality but *also* about the fundamentals. This way it is showed how to merge the theme of learning about others' rationality with the classical theme of learning about fundamentals already developed in standard adaptive learning literature. We will refer to these two connected learning dynamics as the learning determinants of inflation dynamics. Second, the transmission channel from institutional forecasters to private sector is not neutral in that institutional forecasters' estimates are imitated with a noisy, possibly correlated, perturbation. This feature adds a truly macroeconomic flavour in that it reconciles the classical "forecasting the forecast of others' problem", where agents have non negligible impact on aggregate outcomes, with a non-trivial general equilibrium perspective, where an ocean of negligible agents are assumed. In other words, the problem faced by institutional forecasters is not their mere expectation coordination problem because the non-neutrality of the information channel from them to the ocean of agents forming the private sector possibly alters the feedback mechanism. This constitutes what we will call the communication determinant of inflation dynamics. Third, institutional forecasters use a constant gain learning algorithm instead of recursive ordinary least square. Differently from recursive ordinary least square this rule is time invariant and it allows for persistent learning. This work provides also an example on how constant gain cannot only learn a structural change, but it can also trigger it endogenously. Finally, the paper extends analysis of BSE learneability to the case of correlation between institutional forecasters' observational errors. This is a natural extension since institutional forecasters are part of the same public environment, so that they are influenced by same factors. In other words, it is likely, in some extent, that both either have pessimistic perceptions of the other's expectations or have optimistic ones.

1.4 Switching from REE to BSE and viceversa

The main result of the paper is to provide a simple model that exhibit very standard rational expectations behavior *and,* at the same time, it has the potentiality to trigger persistent and endogenous changes in volatility without relying on any Markov switching or additional aggregate shock. All this is basically due to the endogenization of agents' beliefs coordination as described in Gaballo (2009). Here the basic mechanism is implemented and further developed in the context of a simple monetary model in order to defend, in principle, economic relevance of BSE.

The rational expectations equilibrium learneability is proved to generally hold and to be particularly robust to correlation in observational errors. Nevertheless, *even if* institutional forecasters successfully learn about economic fundamentals and rationality of others, one learnable BSE may suddenly arise. Conditions for emergence of a learneable BSE are fulfilled if the transmission channel from institutional forecaster to private sector causes even a very little average amplification of the signal passed by institutional forecasters. The striking feature of this model is that BSE is not alternative to rational expectation equilibrium (REE), but they coexist in a large region of the parameter space. In such a region, real time constant gain learning dynamics selects among them and endogenous and unpredictable switches from one equilibrium to the other can generally arise. I show with numerical simulations how a structural switch from the rational expectations equilibrium to BSE may occur endogenously. Persistent deviations from REE result because, even if agents are all rational and able to consistently estimate fundamentals, they may fail to extract the signal of others' rationality, falling into a coordination failure trap.

1.5 Related literature on excess volatility

Branch and Evans (2007) consider the theme of learning but they focus on evolutionary competition among expectation formation theories in a simple self referential model. Even if central bank is typically the most authoritative forecasting institution several different theories of the same economy are actually employed by agents to forecast. In an evolutionary contest competing theories can coexist because no one is able to perform better than others given their distribution over the population. Using a Lucas-type monetary model, Branch and Evans shape such environment in which different underparameterized theories are available to agents that choose among them on the basis of past performance. They show that Misspecification Equilibria (Branch and Evans 2006a) can arise giving rise to persistent stochastic volatility. Nevertheless, there stochastic volatility is permanent and finally relies on the unavailability of a correctly specified predictor, the only one potentially consistent with REE. This is not the case for the model we are going to present since excess volatility regimes and REE regimes alternates via an endogenous mechanism.

Related is also an extensive stream of literature on excess volatility in asset market returns. We can distinguish mainly four approaches in Macroeconomics. First, Timmermann (1993, 1996), Brennan and Xia (2001) and Cogley and Sargent (2006) among others assume agents implementing Bayesian learning on the dividend process. Those models are not self-referential nature, since agents beliefs do not influence the market outcomes. It is common sense and a simple empirical exercise to test that financial operators actually react to changes in prices as well. Differently, Carceles-Poveda and Giannitsarou (2008), Adam, Marcet and Nicolini (2008a) and Bullard, Evans and Honkapohja (2007) properly takes in to account agents adaptively learning about the prices level. As clarified by Adam, Marcet and Nicolini (2008b) learning about the price level is justified by uncertainty on the marginal agents' expectations, therefore, this scheme considers implicitly the self-referential nature of the model. Later works building on Brock and Hommes (1997, 1998) assumes agents choose among a set of very few sophisticated predictors of the price level relying on relative past performance. Such setting can give rise to complex dynamics and strange attractors. Finally, a recent approach initiated by Allen, Morris and Shin (2006) focuses on the role of high order beliefs of rational short lived agents.

The most important feature of the proposed model in front of quoted literature is that the model is consistent and not alternative to REE. In other words, models above rely on some mechanism that either is exogenously imposed at an aggregate level or persistently alters the volatility regime of the dynamics. Differently, in the model presented below, persistent high volatility regimes endogenously (and unpredictably) arise from a REE regime and viceversa. Moreover the extra noise possibly entering in the equilibrium solution is justified at a micro level, that is, it is not an arbitrary aggregate shock.

2 Model

2.1 A Lucas-type economy

The primary concern of this section is to provide a simple fully microfounded model with the aim to defend, in principle, the economic relevance of behavioral sunspot equilibria. Of course the choice is functional to the scope, so the model is rich enough to embodies standard economic incentives and constraints usually assumed, but also simple enough to have an handy reduced form. Specifically, we will derive a simple Lucas-type monetary model where expectations of current inflation influences actual inflation. It is not a task of this paper to introduce novelties concerning the model in se. To make easier the comparison with closest literature, we will assume the same model with slightly different notation as in Branch and Evans (2007). The key assumptions are the following. We use the convention of a yeoman farmer model (as in Woodford (2003)) provided with a money-in-the-utility function. This is enough to generate a non trivial demand for money responding to classical quantity theory of money without referring to any specification of the financial market. Nevertheless, as first approximation, such demand is assumed to be interest inelastic, so that, dependence of higher order beliefs (forward expectations) is avoided. Finally, we assume a fraction of firms have to set quantities a period before. The latter hypothesis makes expectations about current inflation matter. We now detail the model.

Households. Each farmer produces a differentiated good and sells it in a monopolistically competitive market. In order to introduce price stickiness it is enough to allow for endogenous goods supply. Technology for a representative firm belonging to industry i is given by the following

$$Y_{it} = \psi_t \Omega_t^{-1/(1+\eta)} N_{it}$$

where Ω_{t-1} is the unit labor requirement, ψ_t is a stochastic disturbance and N_{it} is the quantity of labour specific for industry i employed at time t. Let's assume two types of industries; extension to an arbitrary number is straightforward. A representative households solves

$$\max_{\{C_{it},M_{it},N_{it},B_{it}\}} E_0^i \sum_{t=0}^{\infty} \beta^t \frac{C_{it}^{1-\gamma}+M_{it}^{1-\gamma}}{1-\gamma} - \frac{N_{it}^{1+\eta}}{1+\eta}$$
$$\text{s.t.} \quad C_{it}+M_{it}+B_{it} = Y_{it} + \frac{P_{t-1}}{P_t} M_{it-1} + \frac{P_{t-1}}{P_t}\left(1+i_{t-1}\right) B_{it-1}$$

where i_t is the nominal one-period interest rate on debt, E^i is conditional expectation given agent i's information set, B_{it} and M_{it} are respectively bond stock and nominal stock of money held by agent i at time t, and

$$\begin{aligned} C_i &= \left(\int C_{i,j}^{\frac{\theta-1}{\theta}} dj\right)^{\frac{\theta}{\theta-1}}, \\ P_t &= \left(\int P_{j,t}^{1-\theta} dj\right)^{\frac{1}{1-\theta}}, \end{aligned}$$

are CES indexes with $C_{i,j}$ and $P_{j,t}$ being respectively consumption of good j by agent i and price of good j. The aggregate demand Y_t is equal to the integral of individual cost-minimizing demand over agents and goods, formally

$$Y_t = \int \int \left(\frac{P_{j,t}}{P_t} \right)^{-\theta} n_i C_{it} \ di \ dj = \int Y_{it} \ di = \int \left(\frac{P_{j,t}}{P_t} \right)^{-\theta} Y_{j,t} \ dj$$

where Y_{it}, Y_{jt} and n_i are respectively individual aggregate demand over goods j, aggregate demand of good j over individuals i and the fraction of firm type i. The household's first-order conditions can be written as,

$$\begin{aligned} C &: \beta^t C_{it}^{-\gamma} - \lambda_{it} = 0 \\ M &: \beta^t M_{it}^{-\gamma} - \lambda_{it} + E_t^i \lambda_{it+1} \frac{P_t}{P_{t+1}} = 0 \\ B &: -\lambda_{it} + (1 + i_t) \, E_t^i \lambda_{it+1} \frac{P_t}{P_{t+1}} = 0 \\ N &: -\beta^t N_{it}^{\eta} + \lambda_{it} \frac{Y_{it}}{N_{it}} = 0 \end{aligned}$$

where λ_{it} is the Lagrangian multiplier for the budget constraint. We can rewrite condition above solving for λ_{it}. We obtain

$$\begin{aligned} C_{it}^{-\gamma} &= \Omega_{it-1} Y_{it}^{\eta}, \\ C_{it}^{-\gamma} &= M_{it}^{-\gamma} + \beta E_t^i C_{it+1}^{-\gamma} \frac{P_t}{P_{t+1}}, \\ C_{it}^{-\gamma} &= \beta(1 + i_t) E_t^i C_{it+1}^{-\gamma} \frac{P_t}{P_{t+1}}. \end{aligned}$$

These conditions must be satisfied for all i and in all t . In the steady-state $P_{t+1}/P_t = 1$ and $\beta(1 + i_t) = 1$. Combining Euler equations above it is possible to solve for the money-demand function:

$$M_{it} = \left(\frac{i_t}{1 + i_t} \right)^{-\frac{1}{\gamma}} C_{it}.$$

Following Walsh (2003), let $\gamma \to \infty$, so that money demand is interest inelastic and the equilibrium in the money-market requires

$$M_t = Y_t$$

and taking logs of both sides we derive a simple version of the well-known quantity theory of money

$$\ln \overline{M}_t - \ln P_t = \ln Y_t \ , \tag{1}$$

where $\overline{M}_t$ is real money supply. The latter represents the aggregate demand (AD) equation. Notice also such assumption makes consumption and money demand to be independent from (heterogeneous) expectations on future inflation rate. The latter represents the aggregate demand (AD) equation.

Production. Firms set price to maximize profits. Let $P_{i,t}$ be the price in industry i settled by firms taking as given the aggregate price-index P_t. Then a firm's profit function is

$$\Pi \equiv (P_{i,t} - P_t)\, Y_{i,t} = P_{i,t} Y_{i,t} - \frac{\psi_t \Omega_{t-1} Y_{i,t}^{1+\eta} P_t}{C_{i,t}^{-\gamma}}.$$

whose F.O.C. is,

$$\frac{\partial \Pi}{\partial P_{i,t}} + \frac{\partial \Pi}{\partial Y_{i,t}} \frac{\partial Y_{i,t}}{\partial P_{i,t}} = Y_{i,t} + \left(-\theta Y_t \left(\frac{P_t}{P_{i,t}} \right)^{\theta} \right) \left(P_{i,t} - \frac{\psi_t \Omega_{t-1} (1+\eta)\, Y_{i,t}^{\eta} P_t}{C_{i,t}^{-\gamma}} \right) = 0$$

that reduces to

$$\left(\frac{P_{i,t}}{P_t} \right)^{1+\theta\eta} = \frac{\theta}{\theta - 1} \frac{\psi_t \Omega_{t-1} Y_t^{\eta}}{C_{i,t}^{-\gamma}},$$

or, in log form

$$\ln (P_{i,t}) = \ln(P_t) + \frac{\eta}{1+\theta\eta} \ln(Y_t) - \frac{\gamma}{1+\theta\eta} \ln(C_{i,t}) + \frac{1}{1+\theta\eta} \ln \Omega_{t-1} + \ln \left(\frac{\theta}{\theta - 1} \psi_t \right).$$

Following Woodford, assume there is a fraction τ of firms that set prices optimally in every period, while the remaining set their prices one period in advance. Denote $P_{i,f}$,$P_{i,d}$ as the prices of an industry i respectively of type f with flexible prices and d with predetermined prices. Then the log-linearized pricing equations are:

$$\begin{aligned}\ln P_{i,f,t} &= \ln(P_t) + \frac{\eta}{1+\theta\eta}\ln(Y_t) - \frac{\gamma}{1+\theta\eta}\ln(C_{i,t}) + \frac{1}{1+\theta\eta}\ln\Omega_{t-1} + \zeta_t \ , \\ \ln P_{i,d,t} &= E^d_{t-1}\ln P_{i,f,t} \ .\end{aligned}$$

where ζ_t collects the stochastic term in ψ_t. With all agents types evenly distributed across industries it follows that the aggregate price-index can be approximated as,

$$\ln P_t = \tau\left(n\ln P_{1ft} + (1-n)\ln P_{2ft}\right) + (1-\tau)\left(nE^1_{t-1}\ln P_{1ft} + (1-n)\,E^2_{t-1}\ln P_{2ft}\right)$$

and since[2] $E^i_{t-1}\ln P_{ft} = E^i_{t-1}\ln P_t$ we have

$$\begin{aligned}\ln P_t - E_{t-1}\ln P_t &= \frac{\tau}{1-\tau}\left(n\ln P_{1ft} + (1-n)\ln P_{2ft} - \ln P_t\right) \\ &= \frac{\tau}{1-\tau}\left(\left(\frac{\eta-\gamma}{\theta}\right)\ln Y_t + \frac{1}{1+\theta\eta}\ln\Omega_{t-1} + \zeta_t\right)\end{aligned}$$

Therefore, we have the aggregate supply (AS) relation

$$q_t \equiv \ln Y_t - \ln\Omega_{t-1} = \varphi_1\left(\ln P_t - E_{t-1}\ln P_t\right) + \varphi_2\ln\Omega_{t-1} + \varphi_3\zeta_t$$

with

$$\varphi_1 = \frac{\theta(1-\tau)}{\tau(\eta-\gamma)}, \quad \varphi_2 = -\frac{1+\tau\theta\eta}{\tau+\tau\theta\eta}, \quad \varphi_3 = -\frac{(1-\tau)}{\tau},$$

where, for example, $\ln\Omega_{t-1}$ follows a deterministic trend. The AS is a kind of new classical Phillips curve encompassing the one in Lucas (1973), Kydland and Prescott (1977), Sargent (1999), and Woodford (2003).

The economy is represented by equations for aggregate supply (AS) and aggregate

[2] Consider $E^i_{t-1}\ln p_t = \tau E^i_{t-1}\ln p_{ft} + (1-\tau)\,E^i_{t-1}\ln p_{ft}$.

demand (AD):

$$\begin{aligned} AS &: \quad q_t = \varphi_1(p_t - p_t^e) + \varphi_2 \omega_{t-1} + \varphi_3 \zeta_t \\ AD &: \quad q_t = m_t - p_t \end{aligned}$$

where p_t is the log of the price level, p_t^e is the log of expected price formed in $t-1$, m_t is the log of the money supply, q_t is the deviation of the log of real GDP from trend, ζ_t is an i.i.d. zero-mean shock, and ω_t is the log of the unit labor requirement.

Monetary Authority. Assume that the money supply follows

$$m_t - p_t = -(1+\xi)\,(p_t - p_{t-1}) + \delta\omega_{t-1} + u_t \quad \text{with} \quad \xi \geq 0,$$

where u_t is a white noise money supply shock. We are assuming that central bank can observe both p_t and y_t as in Sargent (1987) and Evans and Ramey (2006). Denoting $\pi_t = p_t - p_{t-1}$ we can write the law of motion for the economy in its expectations augmented Phillips curve form

$$\pi_t = \frac{\varphi_1}{1+\varphi_1+\xi}\pi_t^e + \frac{\varphi_2 - \delta}{1+\varphi_1+\xi}\omega_{t-1} + \frac{\varphi_3}{1+\varphi_1+\xi}\zeta_t - u_t$$

or

$$\pi_t = \boldsymbol{\alpha}'\mathbf{z}_{t-1} + \beta\pi_t^e + \nu_t \tag{2}$$

where $\mathbf{z}'_{t-1} \equiv [1 \ \omega_{t-1}]$, $\beta = \frac{\varphi}{1+\varphi+\xi}$, $\boldsymbol{\alpha}' = \begin{bmatrix} 0 & \frac{\varphi_2-\delta}{1+\varphi+\xi} \end{bmatrix}$, $\nu_t \equiv \frac{\varphi_3}{1+\varphi_1+\xi}\zeta_t - u_t$. Note, in particular, that $0 \leq \beta < 1$. The reduced form of this Lucas-type model is very close to the cobweb one. The difference between the two is in the sign of the feedback form expectations: the latter entails a negative feedback, the former a positive one. Differently from the new-Keynesian framework, inflation at time t is affected by expectations at time $t-1$ instead that simultaneous expectations.

Equilibrium. A rational expectations equilibrium (REE) is a stationary sequence

$\{\pi_t\}$ which is a solution to (2) given $\pi_t^e = E_{t-1}\pi_t$, where E_t is the conditional expectations operator. It is well known that (2) has a unique REE and that it is of the form

$$\overline{\pi}_t = (1-\beta)^{-1}\boldsymbol{\alpha}'\mathbf{z}_{t-1} + \nu_t. \tag{3}$$

The REE is a stationary process and cannot explain volatility switching empirically observed. This paper will deal with such unsatisfactory property modifying expectation formation process. As it will be clear later, the reinforcement effect of agents' expectations is a necessary but not sufficient feature for our purposes. Specifically, provided the expectations feedback effect is always damping, the information transmission in the economy will play a key role for emergence of endogenous volatility regime switching. The following section will detail how we are going to modify the rational expectations hypothesis.

2.2 Expectations formation and information diffusion

This section aims to describe how aggregate expectation forms and evolves in time. We will introduce two essential hypothesis in place of the rational expectation hypothesis. First, non-trivial behavioral uncertainty takes place, second, private sector expectations are polarized by two institutional forecasters.

Non-trivial behavioral uncertainty. A general way to model behavioral uncertainty is to assume agents suffer a measurement error in detecting others' simultaneous expectations. Let's denote by $E_{t-1}^i(\cdot)$ agent i's expectation on $(\cdot)$ at time $t-1$. Behavioral uncertainty is entailed formally by

$$\left(E_{t-1}^i E_{t-1}^j \pi_t - E_{t-1}^j \pi_t\right) \equiv v_{i,t-1} \sim \Upsilon\left(0,\delta\right) \tag{4}$$

where $\nu_{i,t-1}$ is a stochastic measurement error drawn from a generic centred distribution function $\Upsilon_i\left(0,\delta\right)$ with zero mean and finite variance δ. In words, agents noisily perceive others' expectation. This is a first (reasonable) departure from REE paradigm in that, form a strict microfounded point of view, REE holds given common knowledge of every agent holds rational expectations. One may want to keep also non-centred distribution, or different type of distributions over the population. This has a sense and it can generate interesting dynamics, nevertheless it doesn't add nothing substantial,but some complexity , to the aim of this paper. As first exercise, we will focus on a unique centred distribution equal for every agent.

Behavioral uncertainty about others' expectations is at the basis of the "forecasting the forecast of others problem" originally posed by Townsend (1983). A following stream of literature investigates how agents can coordinate on rational expectations from a prior disequilibrium (Marcet and Sargent (1989), Sargent (1991), Singleton (1987), Kasa (2000) and Pearlman and Sargent (2004)). All these works consider explicitly a finite number of agents who form expectations independently. Nevertheless, in a general equilibrium perspective, to which the concept of REE refers to, the behavioral uncertainty problem can be just trivial as long as independent idiosyncratic deviations from the rational expectation vanish in the aggregation of an infinite number of agents. As long as individual deviations from the rational expectation are truly random and population is large enough, behavioral uncertainty doesn't add nothing substantial to the individual forecasting problem. In this sense, behavioral uncertainty is non-trivial as long as agents' deviations from REE prescriptions are driven by a common factor whose identification is crucial to optimally solve the individual forecasting problem. In fact, a reasonable doubt that there could exist a non trivial part of agents deviating in a correlated way from REE prescriptions would in turn justifies an individual rational departure from REE prescriptions.

Typically, the coordination of expectations on a particular deviation is yield by the introduction of an exogenous variable working as sunspot. Nevertheless, non-trivial behavioural uncertainty is not consistent with this idea because the emergence of a particular sunspot solution typically requires common knowledge that agents simultaneously believe in such solution. For that no behavioral uncertainty is actually involved in the classical definition of sunspots solutions. Differently, here we want to link the possibility of correlated deviations from REE prescriptions to the fact agents doubt that non-trivial behavioral uncertainty might takes place. To this aim expectation polarization hypothesis is introduced.

Expectations polarization. Let's start from the idea, consistent with statistical learning approach, that forming expectations is a costly activity at least form a cognitive point of view. It is unreasonable to assume that the most part of agents are expert in economics. It is more natural to sooner think they don't have a particular theory on how the economy works. Rather they rely on expectations of some more informed agent like a market leader, or a financial institution that has organizational skills and adequate resources to gather and rationally analyse information. Few institutional forecasters act as focal points for private sector expectations because economies of scale in the "production" of information are typically much stronger than in the production of any other good. The very small number of rating agencies in financial markets is an immediate example of this idea in real economy. In this sense institutional forecasters polarize public expectations.

Let' define formally the structural heterogeneity between an institutional forecasters and private sector. For the sake of simplicity assume there are only two institutional forecasters forming expectations according to

$$E^i_{t-1}x_t \equiv \mathbf{E}[x_t|\Omega^i_{t-1}], \quad \forall i = 1, 2. \tag{5}$$

In words institutional forecasters maintain mathematical expectation of the generic process x_t conditioned to available information up to time $t-1$. Assumption (5) is a formal specification of procedural rationality. It is natural to think professional forecasters are very few because information processing presents strong scale economy effects. We postpone the precise definition of Ω^i_{t-1} until the definition of their learning problem.

Differently, the private sector have the following expectation function specification

$$E^z_{t-1}\pi_t = E^{i_z}_{t-1}\pi_t + v_{z,t-1}, \tag{6}$$

where $E^z_{t-1}(\cdot)$ is nothing else then an imitation correspondence and i_z is the institutional forecaster noisily imitated by agent z belonging to private sector agents set $Z \equiv (0,1)$. Notice that the noise occurs since behavioral uncertainty assumption. If this working hypothesis is reasonable, agents expectations are polarized around few institutional forecasters' forecasts.

Therefore the aggregate expectation is

$$E_{t-1}\pi_t = \int_{z\in Z} E^z_{t-1}\pi_t \; dz = \sum_{i=1,2} \lambda_i E^i_{t-1}\pi_t + \int_{z\in Z} v_{z,t-1} \; dz \ , \quad \sum_{i=1,2} \lambda_i = 1 \tag{7}$$

where $\lambda_i \in (0,1)$ represents the size of the public relying on agent i's expectation. In the present work λ is an exogenous parameter. The extent of agent i's basin of audience, represented by λ_i, measures the average impact of agent i's expectation on the aggregate expectations. It would be very interesting to endogenize it with respect the relative performance of institutional forecasters. This route will be not undertaken in the present work. Nevertheless from here onward we focus on the case of two institutional forecasters polarizing evenly private sector ($\lambda_i = 1/2$). This assumption has a sense given observational error are equal and institutional forecasters both face the same problem, so that $\lambda_i = 1/2$ is by sure a rest point of the replication dynamics driven by institutional forecasters' relative performance.

Equation (7) invalidates the negligibility of agents individual impact in the economy as assumed in general equilibrium perspective. In particular, the impact of each institutional forecaster in the economy is equal and amounts half of the overall aggregate expectation effect. As long as expectations are strongly polarized strategic interaction motives arise among institutional forecasters in expectations formation. We are assuming each agent relies on institutional forecaster's expectations to form his own expectations. Therefore, noisy perceptions about the institutional forecasters' expectations provide information on a common factor embodied in agents' expectations that identifies eventual correlated deviations from REE.

Information diffusion. Figure 1 displays the information diffusion scheme entailed by assumptions above. Two institutional forecasters (red points) affect aggregate expectation calculated over an ocean of agents according to their respective audiences supposed to be equal. The aggregate expectation yields an actual inflation level as implied by (2). Moreover both institutional forecasters have noisy perceptions of the other institutional forecaster's simultaneous expectations. Arrows show flows of information. The two institutional forecasters analyse available data with statistical tools and, on the basis of their estimates, form expectation on future actual inflation.

Three are the key coefficients of the model: β is the feedback of aggregate expectation of current inflation on actual inflation, ρ_v is the correlation coefficient between institutional forecasters' observational errors, and finally γ denotes the covariance between institutional forecasters' expectations and the individual observational error committed by the private sector. The latter measures the non-neutrality of information channel and will be conveniently defined later. In sum, inflation dynamics is affected by learning determinants, that is, how institutional forecasters' expectations evolve in time, and communications determinants, that is what happens to information during the transmission from institutional forecasters to private sector. We will see soon in next section how those three parameters are enough to grasp basic phenomena arising from the interaction of learning (about fundamentals and rationality of others) and institutional communication.

3 From perceived to actual law of motion

3.1 Learning determinants

The emergence of the unique REE depends on the game played by institutional forecasters. Given the power of each institutional forecaster to displace actual inflation away from fundamentals, holding rational expectation is a best expectation if and only if each institutional forecasters believe the other one hold rational expectations. In order to satisfy this requirement institutional forecasters have a double task: learning about fun-

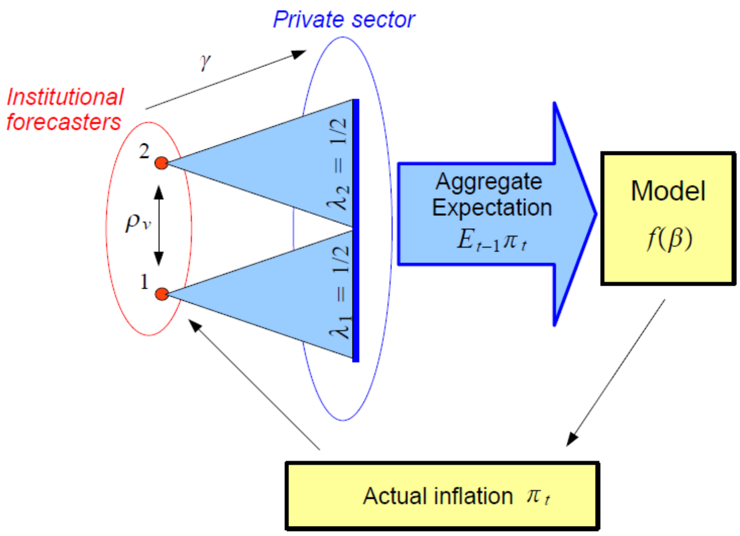

Figure 2: Information diffusion in the economy. Institutional forecasters 1 and 2 analyse data as they become available in time and produce statistically optimal forecasts. Institutional forecasters' expectations polarize evenly private sector expectations. The latter determine, jointly with other exogenous determinants, the actual inflation.

damentals and learning whether or not the other institutional forecaster, and hence a non trivial part of agents, has rational expectations.

Learning about fundamentals: the exogenous long-run component. The REE inflation rate (or fundamental inflation rate) is the long-run component of inflation, denoted by $\overline{\pi}_t$, determined by truly exogenous components. This is the only process compatible with long run equilibrium of agents' forecasts, that is, with rational expectations. Institutional forecasters learn about the fundamental inflation rate regressing a constant and the relevant exogenous variables affecting the economy on actual inflation, namely, in our case, respectively $\mathbf{z}_{t-1}$ on inflation π_t. As standard in adaptive learning literature we assume they hold a correct perceived law of motions encompassing REE form

$$E^i_{t-1}\overline{\pi}_t = \mathbf{a}'_{i,t-1}\mathbf{z}_{t-1}$$

where $\mathbf{a}'_{i,t-1} \equiv [a^c_{i,t-1} \ a^\omega_{i,t-1}]$ are estimated coefficients. Specifically we assume $\mathbf{a}'_{i,t-1}$ is updated recursively in time according to the following constant stochastic gradient (CSG) rule

$$\mathbf{a}_{i,t-1} = \mathbf{a}_{i,t-2} + g_f \ \mathbf{z}_{t-2} \left(\pi_{t-1} - \mathbf{a}'_{i,t-2} \mathbf{z}_{t-2} \right), \tag{8}$$

where g_f is a constant gain smaller than one. Since same information is used, the two estimates asymptotically coincide irrespective of possibly different initial priors, that is $\lim_{t\to\infty} \mathbf{a}_{1,t-1} = \lim_{t\to\infty} \mathbf{a}_{2,t-1}$. Therefore, both have the same forecast of fundamental inflation, that we label $\overline{\pi}_t^e$, with approximation vanishing very soon.

Algorithm (8) is similar to the recursive version of OLS where the estimated correlation matrix is settled equal to the identity matrix and the gain coefficient is fixed[3]. CSG converges to an ergodic distribution centred on the fixed point of the T-map whenever recursive OLS asymptotically converges (to a point). CSG, as any constant gain learning rule, exhibits permanent learning since more weight is given to more recent data. This makes these class of algorithms particularly suitable for learning structural changes. Recursive OLS on the contrary converges at the cost of a huge stickiness of the dynamics after relatively few repetitions. Moreover the CSG algorithm are also derived as optimal solution to a forecast errors variance minimization problem provided agents are "sensitive" to risk in a particular form. For details see Evans, Honkapohja and Williams (2005).

Learning about others' rationality: the endogenous idiosyncratic component. Even in case institutional forecasters correctly estimate fundamental inflation, actual inflation differs from the fundamental one at least for the exogenous stochastic noise ν_t. Nevertheless, because non-trivial behavioral uncertainty is in play, institutional forecasters cannot exclude that such stochastic deviations are due to idiosyncratic departures of aggregate expectation from the rational one. In particular, both institutional

[3] CSG is obtained form recursive constant gain OLS formula

$$\begin{aligned} \mathbf{a}_{t-1} &= \mathbf{a}_{t-2} + \overline{g} R_{t-1}^{-1} \mathbf{z}_{t-2} (\pi_{t-1} - \mathbf{a}'_{i,t-2} \mathbf{z}_{t-2}) \\ R_{t-1} &= R_{t-2} + \overline{g} \left(\mathbf{z}_{t-2} \mathbf{z}'_{t-2} - R_{t-2} \right) \end{aligned}$$

fixing $R_{t-1} = \mathbf{1}$. Therefore, in order to obtain adjustments comparable with constant gain OLS the gain has to be rescaled so that $g = \overline{g}/var(\mathbf{z}_t)$ given $\lim_{t\to\infty} R_{t-1} = var(z_t)$.

forecasters have to understand whether or not deviations from the REE are due to deviations of the other institutional forecasters' expectations from the rational expectation. In other words, they have to assess if the signal about others' expectations is informative about such departures. So, they estimate the optimal weight to give to noisy observations in order to refine their forecasts on actual idiosyncratic inflation deviations from the fundamental. If successful, they extract the signal of others' rationality in real time in that they assess that the noisy information about others' expectations is irrelevant.

They forecast idiosyncratic departure from the fundamental forecasted inflation $\overline{\pi}_t^e$ according to the rule

$$E_{t-1}^1 (\pi_t - \overline{\pi}_t^e) = b_{t-1} \left(E_{t-1}^2 \pi_t + v_{1,t-1} - \overline{\pi}_t^e\right), \tag{9a}$$

$$E_{t-1}^2 (\pi_t - \overline{\pi}_t^e) = c_{t-1} \left(E_{t-1}^1 \pi_t + v_{2,t-1} - \overline{\pi}_t^e\right), \tag{9b}$$

where b_{t-1} and c_{t-1} are recursively estimated with CSG

$$\begin{aligned} b_{t-1} &= b_{t-2} + g_d \left(E_{t-1}^2 \pi_t + v_{1,t-1} - \overline{\pi}_t^e\right) (\pi_{t-1} - E_{t-2}^1 \pi_{t-1}), \\ c_{t-1} &= c_{t-2} + g_d \left(E_{t-1}^1 \pi_t + v_{2,t-1} - \overline{\pi}_t^e\right) (\pi_{t-1} - E_{t-2}^2 \pi_{t-1}), \end{aligned}$$

where g_d is the updating gain, $(\pi_t - E_{t-2}^i \pi_t)$ is the forecast error and $\left(E_{t-1}^j \pi_t + v_{i,t-1} - \overline{\pi}_t^e\right)$ is the noisy observed displacements of others' expectations from the estimated fundamental one.

If (b_{t-1}, c_{t-1}) asymptotically converge to zero institutional forecasters will forecast the fundamental value, so that aggregate expectation will be a rational expectations. In other words, if all institutional forecasters are rational they would not need to condition their expectations on noisy observations of the other one's simultaneous expectations. But if this is not the case considering noisy observations actually improves the accuracy of forecasts. Learning is valuable exactly because this form of behavioral uncertainty is introduced. In this case, CSG has the advantages of showing convergence to equilibria and, at the same time, the possibility of endogenous and unpredictable shifts from the REE to a BSE. For details about this learning scheme the interested reader can refer to Gaballo (2009).

3.2 Communication determinants

Non-neutrality of the information channel. The process $v_{z,t-1}$ features the effect of information transmission from institutional forecasters to private sector, formally measured by $\int_{z \in Z} v_{z,t-1}\ dz$ in (7). It is convenient to express $v_{z,t-1}$ in the following way

$$v_{z,t} = \gamma(E_{t-1}^{i_z}\pi_t - \overline{\pi}_t) + (1-\gamma)\,\epsilon_{z,t} \tag{10a}$$

where $\epsilon_{z,t}$ is a i.i.d. shock distributed according $\Upsilon\left(0, \left(1-\gamma^2\mathbf{E}(E_{t-1}^{i_z}\pi_t - \overline{\pi}_t)^2\right)/(1-\gamma)^2\right)$ so that overall variance is simply $\mathbf{E}\left(v_z^2\right) = \delta$. The specific forms of observational errors as maintained by (10a) do not add nothing substantial in the general framework. The coefficient γ controls for the covariance between this observational error, and the estimated distance of the actual inflation from the estimated fundamental one. Note that the latter is a proxy for the amount of behavioural uncertainty in the economy. In other words, this specification takes account of the idea that the public receives a biased information whose idiosyncratic component is possibly further amplified or dumped in the transmission. Finally, from aggregation over Z we have

$$\int_{z \in Z} v_z\ dz = \frac{\gamma}{2}\left(\sum_{i=1,2}(E_{t-1}^{i}\pi_t - \overline{\pi}_t^e)\right), \tag{11}$$

so that the aggregate expectation $E_{t-1}\pi_t$, is now equal to

$$E_{t-1}\pi_t = \overline{\pi}_t^e + \frac{(1+\gamma)}{2}\sum_{i=1,2}\left(E_{t-1}^{i}\pi_t - \overline{\pi}_t^e\right), \tag{12}$$

that is, it depends on both institutional forecasters' expectations of displacements of actual inflation rate from the estimated fundamental one *and* on the neutrality of the information channel measured by γ.

3.3 The actual law of motion

From (9) it is simple to check that institutional forecasters' expectations can be expressed as linear function of the fundamental price and observational errors. Formally we have

$$E^1_{t-1}\pi_t = \overline{\pi}^e_t + \frac{b_{t-1}c_{t-1}}{1-b_{t-1}c_{t-1}}v_{2,t-1} + \frac{b_{t-1}}{1-b_{t-1}c_{t-1}}v_{1,t-1} \quad (13a)$$

$$E^2_{t-1}\pi_t = \overline{\pi}^e_t + \frac{b_{t-1}c_{t-1}}{1-b_{t-1}c_{t-1}}v_{1,t-1} + \frac{c_{t-1}}{1-b_{t-1}c_{t-1}}v_{2,t-1} \quad (13b)$$

provided $bc \neq 1$. Processes (13) cannot be inferred by agents since they cannot distinguish observational errors. According to (2), (12) and (13) makes the actual law of motion to move according to the following process

$$\pi_t = \boldsymbol{\alpha}'\mathbf{z}_{t-1} + \beta\overline{\pi}^e_t + \frac{\beta^*}{2}\left(\frac{b_{t-1}\left(1+c_{t-1}\right)}{1-b_{t-1}c_{t-1}}v_{1,t-1} + \frac{c_{t-1}(b_{t-1}+1)}{1-b_{t-1}c_{t-1}}v_{2,t-1}\right) + \nu_t \quad (14)$$

where $\beta^* \equiv \beta(1+\gamma)$. Notice that $\pi_t = \boldsymbol{\alpha}'\mathbf{z}_{t-1} + \beta\overline{\pi}^e_t$ if: *i*) agents are not uncertain about others' behavior, that is $v_{1,t-1} = 0$ and $v_{2,t-1} = 0$, or *ii*) agents hold the rational expectation, that is, $b_{t-1} = 0$ and $c_{t-1} = 0$, or *iii*) expectations have a zero impact on the actual course given $\beta^* = 0$. In such cases the problem reduces to the simple one extensively studied in classical adaptive learning literature in relation to the cobweb reduced form. Otherwise inflation will exhibit endogenous excess volatility around the estimated inflation due by the stochastic term originated by non linear combination of observational errors and possibly further amplified (or dumped) by the transmission channel term impacting through β^*.

4 Equilibria and Learneability

4.1 Equilibria

Equilibria are such that institutional forecasters' forecast errors are orthogonal to available information, namely to exogenous variables time series and noisy perceptions of others' expectations. Formally they have to solve the following system

$$\begin{aligned}
\mathbf{E}[\mathbf{z}_{t-1}(\pi_t - T'_{\mathbf{a}}\mathbf{z}_{t-1})] &= 0 \qquad (15)\\
\mathbf{E}[\left(E^2_{t-1}\pi_t + v_{1,t-1} - \overline{\pi}^e_t\right)\left(\pi_t - \overline{\pi}^e_t - T_b\left(E^2_{t-1}\pi_t + v_{1,t-1} - \overline{\pi}^e_t\right)\right)] &= 0 \qquad (16)\\
\mathbf{E}[\left(E^1_{t-1}\pi_t + v_{2,t-1} - \overline{\pi}^e_t\right)\left(\pi_t - \overline{\pi}^e_t - T_c\left(E^1_{t-1}\pi_t + v_{2,t-1} - \overline{\pi}^e_t\right)\right)] &= 0 \qquad (17)
\end{aligned}$$

where $T_{\cdot}$ map gives the coefficients of the linear forecast rule yielding local minima of the mean square error variance conditioned on the available information set. For a technical reference on projections and convergence properties of adaptive learning algorithms used in what follows, see Marcet and Sargent (1989), Evans and Honkapohja (2001).

Proposition 1 *T-map takes the form*

$$\begin{aligned}
T_{\mathbf{a}}(\mathbf{a}) &= \boldsymbol{\alpha} + \beta\mathbf{a}\\
T_b(b,c) &= \frac{\beta^*}{2}\left(\frac{b(1+c)(1+c\rho_v) + c(1+b)(c+\rho_v)}{1+c^2(1+2\rho_v)}\right)\\
T_c(b,c) &= \frac{\beta^*}{2}\left(\frac{b(1+c)(b+\rho_v) + c(1+b)(1+b\rho_v)}{1+b^2(1+2\rho_v)}\right)
\end{aligned}$$

Proof. Keep in mind that observational errors $v_{i,t-1}$ have zero mean and they are uncorrelated with exogenous variable $\mathbf{z}_{t-1}$, that is $E[v_{i,t-1}\mathbf{z}_{t-1}] = \mathbf{0}$. Spelling out conditions for $T_{\mathbf{a}}$ and T_b (T_c is mirror like),we have respectively

$$T_{\mathbf{a}} : \boldsymbol{\alpha}'\ E[\mathbf{z}'_{t-1}\mathbf{z}_{t-1}] + \beta\ \mathbf{a}'\ E[\mathbf{z}'_{t-1}\mathbf{z}_{t-1}] - T'_{\mathbf{a}}\ E[\mathbf{z}'_{t-1}\mathbf{z}_{t-1}] = 0$$

and

$$T_b : \frac{\beta^*}{2}\left(\frac{b\left(1+c\right)}{\left(1-bc\right)^2}\delta+\frac{c^2(b+1)}{\left(1-bc\right)^2}\delta+\frac{c(b+1)+cb\left(1+c\right)}{(1-bc)^2}\delta\rho_v\right)+$$
$$-T_b(\frac{1}{\left(1-bc\right)^2}\delta+\frac{c^2}{\left(1-bc\right)^2}\delta+2\frac{c}{\left(1-bc\right)^2}\delta\rho_v)=0$$

Finally the projected T map for $\mathbf{a}$, b and c is given by

$$\begin{aligned} T_{\mathbf{a}} &= \boldsymbol{\alpha}+\beta\mathbf{a} \\ T_b &= \frac{\beta^*}{2}\left(\frac{b\left(1+c\right)\left(1+c\rho_v\right)+c(b+1)\left(c+\rho_v\right)}{1+c^2+2c\rho_v}\right), \\ T_c &= \frac{\beta^*}{2}\left(\frac{b\left(1+c\right)\left(b+\rho_v\right)+c(b+1)\left(1+b\rho_v\right)}{1+b^2+2b\rho_v}\right). \end{aligned}$$

■

Notice T-map depends on error variances ratio $\varepsilon_1/\varepsilon_2$ and not at all on the extent of them. Moreover if $\varepsilon_1=\varepsilon_2$ errors variances simply disappear from equations.

Since the inflation process is endogenously determined by agents' forecasts, the T-map depends on the coefficients of the forecast rules. Therefore, fixed points of the T-map are the values for which professional forecasters do not commit systematic error given available information.

Definition 2 *Equilibria obtain as fix points of the T map for $T_{\mathbf{a}}\left(\widehat{\mathbf{a}}\right)=\widehat{\mathbf{a}}$, $T_b\left(\widehat{b},\widehat{c}\right)=\widehat{b}$ and $T_c\left(\widehat{b},\widehat{c}\right)=\widehat{c}$.*

Now it is possible to state the following.

Proposition 3 *Equilibria of the system are:*

i) a REE $(\widehat{\mathbf{a}}',\widehat{b},\widehat{c})=((1-\beta)^{-1}\boldsymbol{\alpha}',0,0)$,

ii) an high BSE $(\widehat{\mathbf{a}}',\widehat{b},\widehat{c})=((1-\beta)^{-1}\boldsymbol{\alpha}',\frac{\beta^*-(2-\beta^*)\rho_v+2\sqrt{(\beta^*-1)(1-\rho_v^2)}}{2-\beta^*(1+\rho_v)},\frac{\beta^*-(2-\beta^*)\rho_v+2\sqrt{(\beta^*-1)(1-\rho_v^2)}}{2-\beta^*(1+\rho_v)})$,

iii) a low BSE $(\widehat{\mathbf{a}}',\widehat{b},\widehat{c})=((1-\beta)^{-1}\boldsymbol{\alpha}',\frac{\beta^*-(2-\beta^*)\rho_v-2\sqrt{(\beta^*-1)(1-\rho_v^2)}}{2-\beta^*(1+\rho_v)},\frac{\beta^*-(2-\beta^*)\rho_v-2\sqrt{(\beta^*-1)(1-\rho_v^2)}}{2-\beta^*(1+\rho_v)})$.

Proof. Equilibria are given by the system:

$$\widehat{\mathbf{a}}' = \boldsymbol{\alpha} + \beta\widehat{\mathbf{a}}' \tag{18a}$$

$$\widehat{b} = \frac{(\beta^*/2)\,\widehat{c}(\widehat{c}+\rho_v)}{(1-(\beta^*/2)\,(1-\rho_v)\,)\widehat{c}^2+((2-\beta^*)\rho_v-\beta^*/2)\widehat{c}+(1-(\beta^*/2)\,)} \tag{18b}$$

$$\widehat{c} = \frac{(\beta^*/2)\,\widehat{b}(\widehat{b}+\rho_v)}{(1-(\beta^*/2)\,(1-\rho_v)\,)\widehat{b}^2+((2-\beta^*)\rho_v-\beta^*/2)\widehat{b}+(1-(\beta^*/2)\,)} \tag{18c}$$

assuming $bc \neq 1$. It is easily proved by substitution that the fundamental rational expectation solution it is always a rest point of the T-map. Other non fundamental $\widehat{b}$ and $\widehat{c}$ equilibria values are in correspondence of $\widehat{b} = \widehat{c}$ and result as solutions to

$$\widehat{c}\left(\widehat{c}^2\left((1-\beta^*/2)-(\beta^*/2)\,\rho_v\right)-\left(\beta^*-(2-\beta^*)\,\rho_v\right)\widehat{c}+(1-\beta^*/2)-(\beta^*/2)\,\rho_v\right)=0 \tag{19}$$

featuring respectively the high BSE values (b_+, c_+) and the low BSE values (b_-, c_-) where

$$c_+ = b_+ = \frac{\beta^*-(2-\beta^*)\,\rho_v+2\sqrt{(\beta^*-1)\,(1-\rho_v^2)}}{2-\beta^*\,(1+\rho_v)} \tag{20}$$

$$c_- = b_- = \frac{\beta^*-(2-\beta^*)\,\rho_v-2\sqrt{(\beta^*-1)\,(1-\rho_v^2)}}{2-\beta^*\,(1+\rho_v)} \tag{21}$$

exist whenever $\beta^* \geq 1$. ■

The figure below plots T_b for four different calibrations. Given the symmetric nature of the problem we are analyzing, BSEs are at the intersection of T_b with bisector. Line a is obtained for $\beta^* = 0.8$ and $\rho_v = 0$. In such a case the unique intersection is at the REE values $\widehat{b} = \widehat{c} = 0$. As β^* goes up to one (line b), two BSE emerge such that not trivial values of $\widehat{b}$ and $\widehat{c}$ exist such that forecast errors are uncorrelated with available information. Ceteris paribus increasing values of ρ_v (line c) makes the BSE with smaller values closer to REE values and the high one being further away. Finally, extreme calibration as the one showed by line d yields negative BSEs; the low one is shown in the picture.

The arising of equilibria different from the REE is due to non-linearity of the T-map induced by the non-linear constraint linking observational errors as they appears in (9).

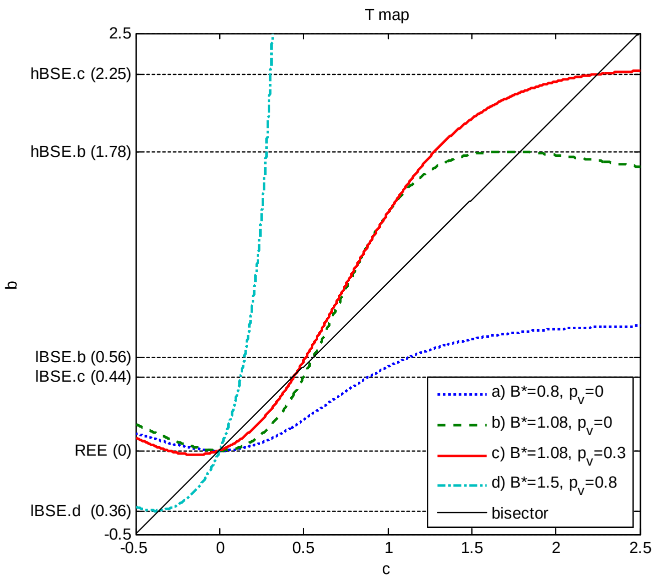

Figure 3: Tmap representation for different calibration. Equilibria are at intersection with Tmap with the bisector. For values of β^* bigger than one two BSEs arise besides REE.

In this respect, BSEs are kind of limited-informed rational expectation (Sargent 1991), because agents cannot observe all the stochastic components of the actual law of motion separately. The non-linear link between observational errors generates externalities to the individual forecasting problem. In fact, BSE are kind of coordination failures in that once achieved variance of observational errors transmits persistently to the course of actual output gap through aggregate expectation, making the overall forecast error variance higher then the REE one, even if locally minimal. Such equilibria do not require any external coordinating mechanism or common knowledge assumption. They arise as the result of endogenous coordination among non cooperative agents. In that respect they are qualitatively different from classical sunspot equilibria. For further details about BSE see Gaballo (2009).

4.2 Learneability

This section explores leaneability of REE and the possibility of adaptive learners being stuck in a BSE, that is whether or not BSE are learneable . The concept of learneability refers to the nature, stable or unstable of the learning dynamics around the equilibria computed above under a recursive least square algorithm .

Definition 4 *An equilibrium* $(\widehat{\mathbf{a}},\widehat{b},\widehat{c})$ *is locally learnable under recursive least square (RLS) algorithm if and only if there exist some neighborhood* $\Im(\widehat{\mathbf{a}},\widehat{b},\widehat{c})$ *of* $(\widehat{\mathbf{a}},\widehat{b},\widehat{c})$ *such that for each initial condition* $(\mathbf{a}_0,b_0,c_0) \in \Im(\widehat{\mathbf{a}},\widehat{b},\widehat{c})$ *the estimates converge almost surely to the equilibrium, that is* $\lim_{t\to\infty}(\mathbf{a}_{t-1},b_{t-1},c_{t-1}) \overset{a.s.}{=} \left(\widehat{\mathbf{a}},\widehat{b},\widehat{c}\right)$.

To check learneability one need to investigate the Jacobian of the T-map. If the matrix of all partial derivative of T-map in the equilibrium has all eigenvalues lie inside the unit circle, we can say the equilibrium to be stable under learning (Marcet and Sargent 1989, Evans Honkapohja 2001). The Jacobian for T-map takes the form

$$JT(\mathbf{a},b,c) = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & \beta & 0 & 0 \\ 0 & 0 & \frac{dT_b(b,c)}{db} & \frac{dT_b(b,c)}{dc} \\ 0 & 0 & \frac{dT_b(b,c)}{dc} & \frac{dT_c(b,c)}{dc} \end{pmatrix}$$

where

$$\frac{\mathrm{d}T_b(b,c)}{\mathrm{d}b} = \frac{\beta^*}{2}\frac{(1+c)(1+c\rho_v)+c(c+\rho_v)}{1+c^2+2c\rho_v}, \tag{22a}$$

$$\frac{\mathrm{d}T_b(b,c)}{\mathrm{d}c} = \frac{\beta^*}{2}\frac{b(1+c\rho_v)+b\rho_v\,(1/2)\,(1+c)+(c+\rho_v)(1+b)}{1+c^2+2c\rho_v} + \\ - \frac{2(c+\rho_v)(b(1+c\rho_v)(1+c)+c(c+\rho_v)(1+b))}{1+c^2+2c\rho_v}, \tag{22b}$$

$$\frac{\mathrm{d}T_c(b,c)}{\mathrm{d}b} = \frac{\beta^*}{2}\frac{c(1+b\rho_v)+c\rho_v\,(1/2)\,(1+b)+(b+\rho_v)(1+c)}{1+b^2+2b\rho_v} + \\ - \frac{2(b+\rho_v)(c(1+b\rho_v)(1+b)+b(b+\rho_v)(1+c))}{1+b^2+2b\rho_v}, \tag{22c}$$

$$\frac{\mathrm{d}T_c(b,c)}{\mathrm{d}c} = \frac{\beta^*}{2}\frac{(1+b)(1+b\rho_v)+b(b+\rho_v)}{1+b^2+2b\rho_v}. \tag{22d}$$

To analyse learneability of equilibria we have to investigate the sign of eigenvalues of the matrix $K \equiv JT - I$ (where I is the identity matrix) in the equilibrium values $\widehat{\mathbf{a}}$ and

$\widehat{c} = \widehat{b}$ given by

$$K_{(\widehat{\mathbf{a}},\widehat{b},\widehat{c})} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \beta - 1 & 0 & 0 \\ 0 & 0 & \left[\frac{dT_b(b,c)}{db}\right]_{(\widehat{b},\widehat{c})} - 1 & \left[\frac{dT_b(b,c)}{dc}\right]_{(\widehat{b},\widehat{c})} \\ 0 & 0 & \left[\frac{dT_c(b,c)}{db}\right]_{(\widehat{b},\widehat{c})} & \left[\frac{dT_c(b,c)}{dc}\right]_{(\widehat{b},\widehat{c})} - 1 \end{pmatrix}, \tag{23}$$

with

$$\begin{aligned}
\left[\frac{dT_b(b,c)}{db}\right]_{(\widehat{b},\widehat{c})} - 1 &= \frac{((\beta^*/2)(1+\rho_v) - 1)\,\widehat{b}^2 + ((\beta^*/2)(1+2\rho_v) - 2\rho_v))\,\widehat{b} + (\beta^*/2) - 1}{1 + \widehat{b}^2 + 2\widehat{b}\rho_v}, \\
\left[\frac{dT_b(b,c)}{dc}\right]_{(\widehat{b},\widehat{c})} &= (\beta^*/2)\,\frac{(2\rho_v^2 - 1)\widehat{b}^3 + 3\rho_v^2\widehat{b} + 3\widehat{b} + \rho_v}{\left(1 + \widehat{b}^2 + 2\widehat{b}\rho_v\right)^2}, \\
\left[\frac{dT_c(b,c)}{dc}\right]_{(\widehat{b},\widehat{c})} &= \left[\frac{dT_b(b,c)}{db}\right]_{(\widehat{b},\widehat{c})} \quad \text{and} \quad \left[\frac{dT_c(b,c)}{db}\right]_{(\widehat{b},\widehat{c})} = \left[\frac{dT_b(b,c)}{dc}\right]_{(\widehat{b},\widehat{c})}.
\end{aligned}$$

A certain equilibrium $(\widehat{\mathbf{a}},\widehat{b},\widehat{c})$ is learnable if and only if the matrix $K_{(\widehat{\mathbf{a}},\widehat{b},\widehat{c})}$ has all negative eigenvalues. Figure 4 below shows numerical analysis for the whole parameter range[4] spanned by β^* and ρ_v. Keep in mind that a necessary condition for learneability of equilibria is always $\beta < 1$. We presume it in the following discussion.

As is evident from inspection of the plot REE is the only learneable equilibrium in the region $\beta^* < 1$. In the white area a learneable high BSE (hBSE) arises besides a REE. This area is the most interesting in that it partially includes most realistic calibration values for the Lucas-type monetary model. Notice that whenever a hBSE exists it is not the unique learneable equilibrium. For such values the learning mechanism selects

[4] From quite immediate application of a proposition proved in Gaballo (2009), REE solution $(\widehat{\mathbf{a}},\widehat{b},\widehat{c}) = (0, \frac{\alpha}{1-\beta}, 0, 0)$ is learneable whenever

$$\beta < 1, \tag{24}$$

$$\beta^* \leq \frac{2}{1+\rho_v} \quad \text{with } \rho_v \geq 0, \tag{25}$$

$$\beta^* \leq \frac{2}{1-\rho_v} \quad \text{with } \rho_v < 0. \tag{26}$$

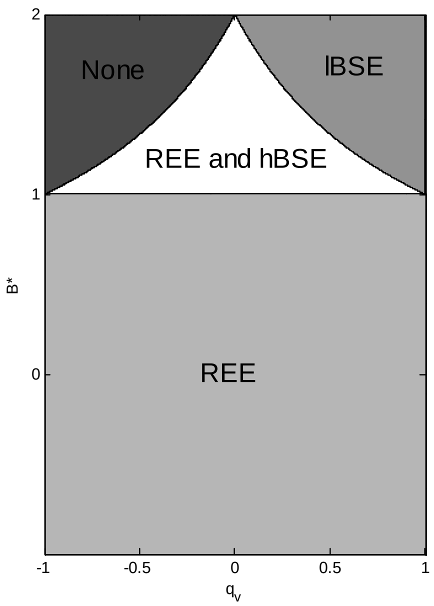

Figure 4: Numerical learneability analysis in the whole parameter space. The space is partitioned in four regions exhibiting different learneability properties. In the white one REE and the high BSE are both learneable and learning dynamics select among them. In the light grey only REE is learneable whereas in the dark grey only the low BSE is learneable. In the black area none learneable equilibria are present.

between REE and hBSE. How this happens will be explained in detail later, when we will present numerical simulation of the dynamic system. REE and hBSE are both learneable for lower values of β^* as ρ_v increases in modulus. Specifically, as ρ_v approach unity for sufficiently high value of β^* the low BSE (lBSE) becomes learneable and both REE and hBSE are no longer. This is a standard property of non linear dynamics: given the system has three equilibria, either the most distant two are dynamically stable or only the one in the middle is dynamically stable (refer to figure 3). On the other hand as ρ_v decreases for sufficiently high value of β^* the system presents no learneable equilibria.

Whenever learneable BSEs exist, distances between equilibria measured on the bisector line in figure 3 are indicative of the size of basins of attraction. In particular, at least in the range considered, the T-map behaves like a cubic yielding three dynamic equilibria. As usual, the middle equilibrium either is unstable and works as threshold between the basins of attraction of the other (stable) two or is the unique stable equilibria with a basin of attraction lying between equilibria at the extremes. As example for calibration b and c in figure 3, the REE and the high BSE basins of attraction are divided by the low BSE. In particular as β^* increases REE basin of attraction shrinks, whereas the high BSE one enhances.

Finally, notice that necessary condition for arising and learneability of BSEs in this model is that $\gamma > 0$, that is, private sector agents commits observational errors that are correlated in average with the idiosyncratic departures of actual price from the fundamental one forecasted by institutional forecasters. In that respect transmission of information in the economy plays an essential role. What matters is that forecasted idiosyncratic departure from the fundamental rate of inflation are amplified by private sector overreaction. Other behavioral schemes of the private sector can be actually implemented to obtain the same, or more complex, dynamics.

4.3 Excess volatility

Equilibria with $\left(\widehat{b}, \widehat{c}\right) \neq (0,0)$ includes extra stochastic variables, namely observational errors, with no economic content in agents' expectation function. BSE present, as any sunspot solution, a volatility higher than the fundamental solution. The extent of theoretical excess variance is measured in equilibrium $\left(\widehat{b} = \widehat{c}\right)$ by

$$2\left(\frac{\beta^*\widehat{b}}{2\left(1-\widehat{b}\right)}\right)^2 \left(\varrho + \rho_v\right),$$

and it is increasing in β, δ and ρ_v and decreasing in $\widehat{b}$ for $\widehat{b} > 1$.

The picture below plots excess variance yield by learneable BSE in terms of observational error variance for values $\beta^* \in (1,2)$ ("5" stays for "5 and more"). For values close to unity excess variance is really high but it decreases very soon. The most part of the relevant region exhibits an excess volatility between one and four times the variance of observational errors. In the region for which REE and the high BSE are both learneable we

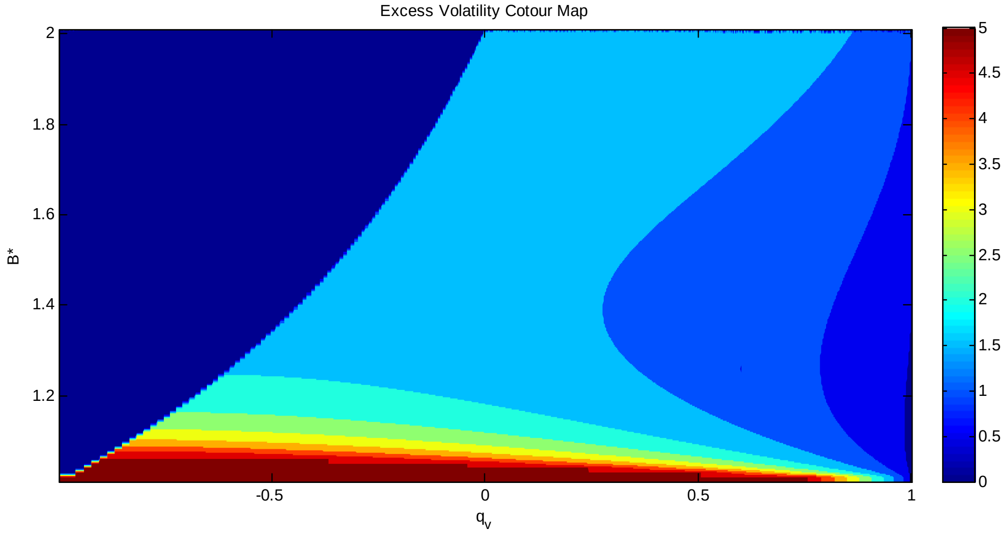

Figure 5: Numerical analysis of excess volatility. The picture shows the size of excess volatility obtained for values $\beta^* \in (1, 2)$ for which learneable BSEs arise. The unit of measure of the scale is the variance of observational errors. ("5" stays for "5 and more").

can have different volatility regimes (an high volatility one being in correspondence of the high BSE) depending on the equilibrium selected by the learning algorithm. Next section we finally explain and show how unpredictable and endogenous switching of volatility regimes can be triggered by constant gain algorithm.

5 Constant gain learning simulation

The simulations proposed in this section provides examples of endogenous and unpredictable changes in volatility regimes. We chose calibrations such that analytical results can be contrasted with experiments. All simulations are generated with the following parameter setting: $\beta = 0.8$, $\delta = \varphi_2$, $\delta = 0.1$. The exogenous shocks are all Gaussian white noises with unit variance. In all figures the following conventions hold. In the upper box is displayed the dynamics of the two coefficients, b_t and c_t. Whenever low BSE values serve as divide between REE and high BSE basins of attraction, these are indicated by a dotted line in the upper box. The lower box shows the corresponding dynamics of both

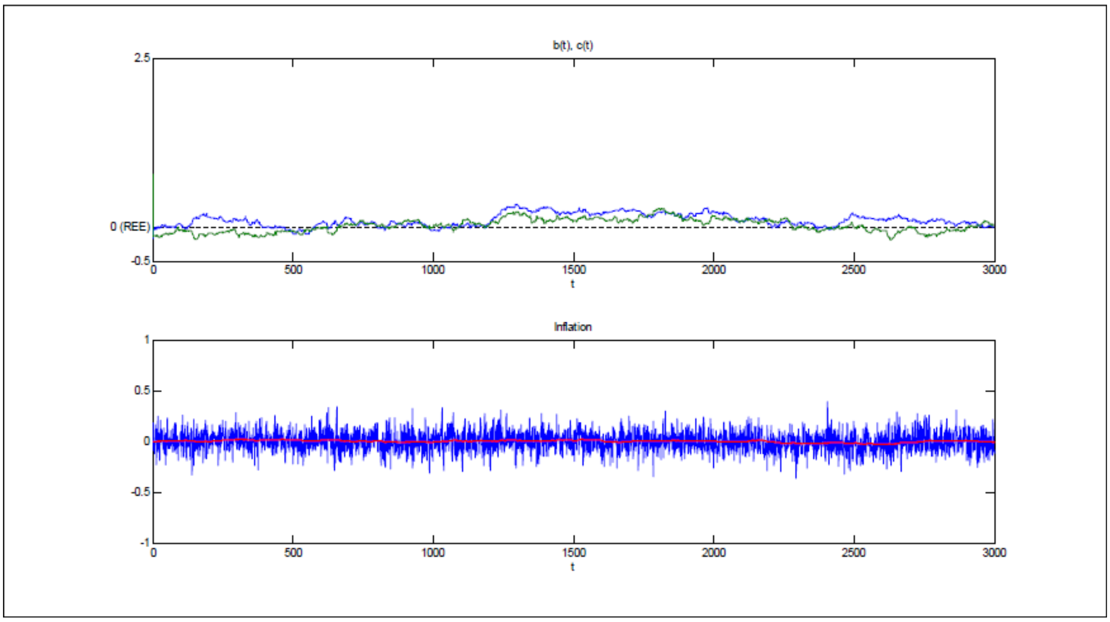

Figure 6: Benchmark case. Convergence to REE ($\beta = 0.8$, $\gamma = 0$, $\rho_v = 0$). Line a). in figure 3.

actual inflation π_t and agents' estimated fundamental inflation (flatter line). The first four figures are generated with the same series of errors and with same initial conditions closely set around REE value.

Figure 6 displays the benchmark case, that is, convergence in distribution to REE values for $\rho_v = \gamma = 0$. The gain is settled $g_d = g_f = \delta^{-1}/110$. The factor δ^{-1} has been included in the gain so that the adjustments of both b_t and c_t are substantially equal to the ones obtained with constant gain OLS around REE values for $\overline{g} = 110.$. Notice how constant gain learning generate continuous small displacements away from REE values. Nevertheless such diplacements are temporary escapes and do not substantially affect the variance of actual inflation process. In the second box one can appreciate the near-natural REE variance and how the estimate of fundamental inflation soon approaches the REE value.

In figure 6 the calibration of figure 1 is modified only in that $\gamma = 0.28$ (so that $\beta^* = 1.08$). For such values one learneable high BSE arise for $b = c = 1.78$. Up to 1300 periods the dynamics is roughly the same, but how estimates approach low BSE values,

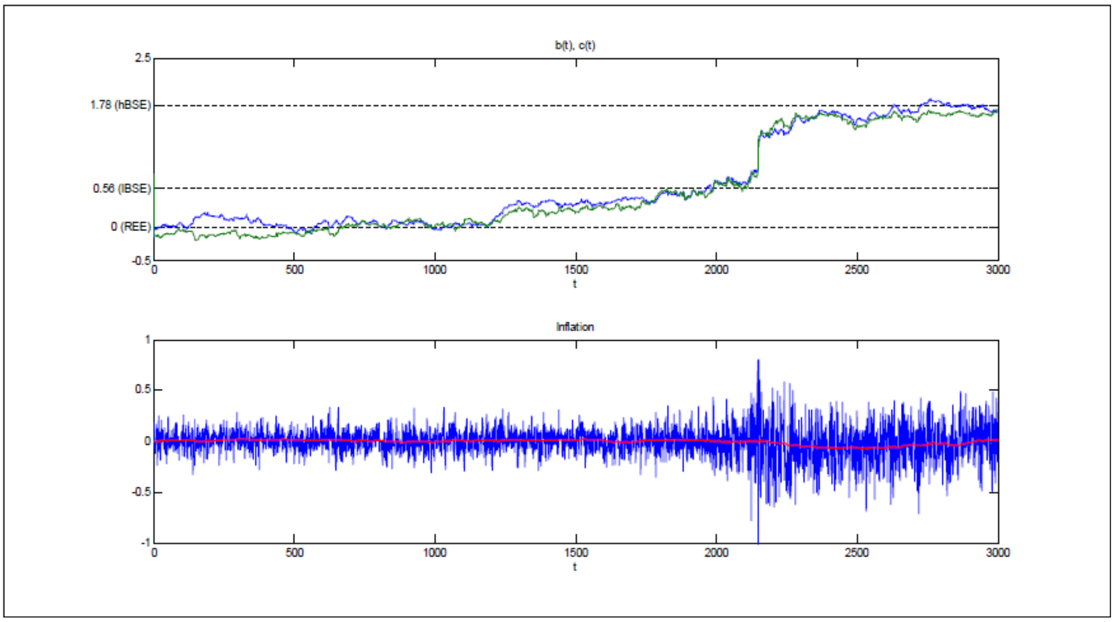

Figure 7: From REE to hBSE ($\beta = 0.8$, $\gamma = 0.28$, $\rho_v = 0$). Line b). in figure 3.

the dynamics changes dramatically. In particular as estimates overcome low BSE values (around 2000 periods) the dynamics enters in the basin of attraction of the high BSE making estimates to converge in distribution to it. This endogenous structural change affect in a persistent and substantial way actual inflation variance. The resulting excess variance is about three times REE variance. Notice how excess volatility generated by high BSE affects volatility of the estimated fundamental inflation too, contributing to the overall variance of actual inflation. This effect is more evident in next and last pictures.

Figure 8 is generated with same setting introducing a small correlation between observational errors $\rho_v = 0.3$. The effect of this type of correlation is in a earlier jump to the correspondent learneable high BSE. This is not surprising since for increasing positive values of ρ_v the corresponding high BSE values increase and low BSE ones decrease. This means that high BSE basin of attraction enhances and REE basin shrinks, so that jumps from REE to high BSE is more likely to happen. As already noted, it is possible to appreciate this feature contrasting line b and c in figure 3.

Figure 9 provides an example of convergence to the low BSE. This occurs for quite extreme and careful calibration in that low BSE basin of attraction is quite narrow given

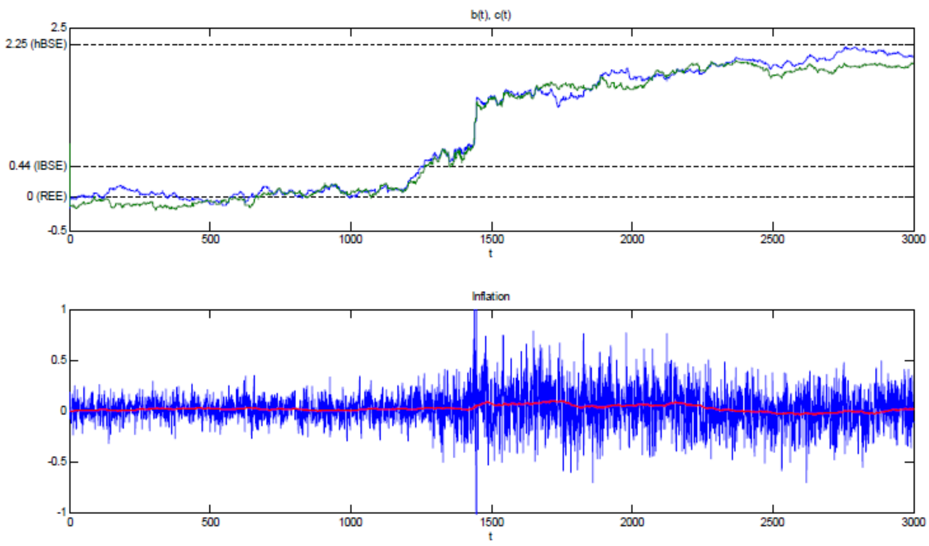

Figure 8: From REE to hBSE ($\beta = 0.8$, $\gamma = 0.28$, $\rho_v = 0.3$). Line c). in figure 3.

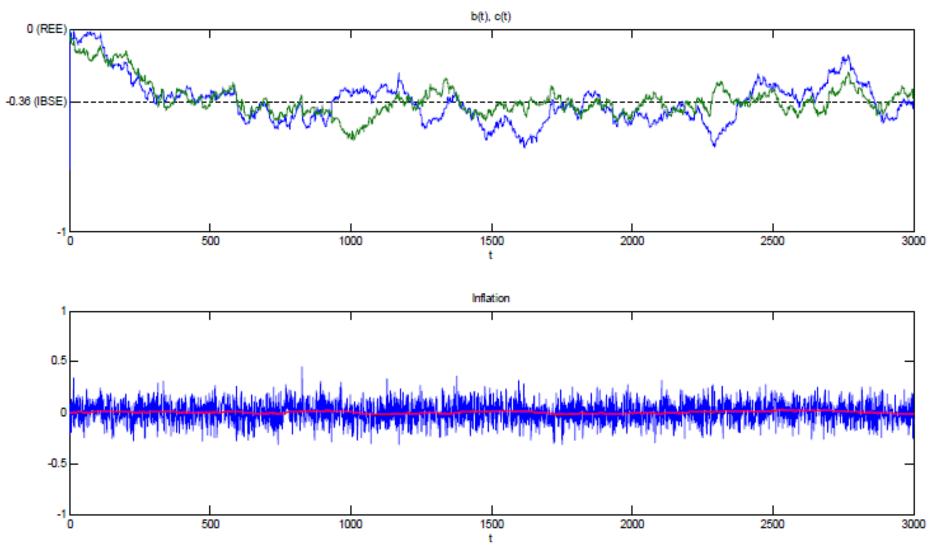

Figure 9: Convergence to lBSE ($\beta = 0.8$, $\gamma = 0.7$, $\rho_v = 0.8$). Line d). in figure 3.

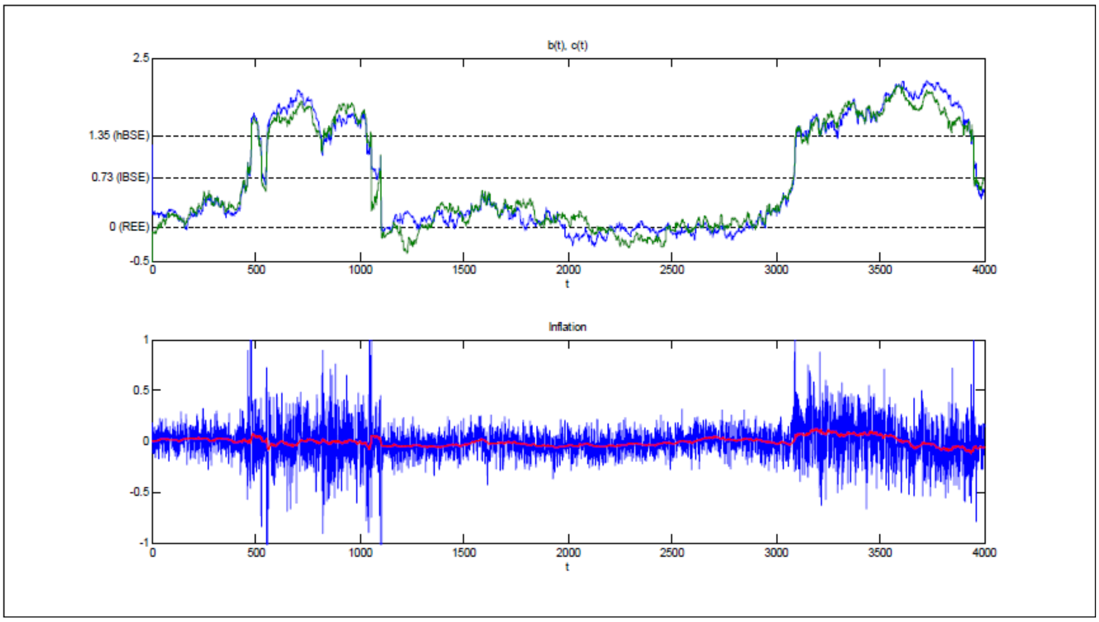

Figure 10: From REE to hBSE and back two times ($\beta = 0.8$, $\gamma = 0.21$, $\rho_v = 0.4$).

the closeness of low BSE values to REE ones. The one displayed is obtained for $\gamma = 0.7$ and $\rho_v = 0.8$. As evident the contribute to overall actual inflation variance is almost negligible. Finally last picture shows how with appropriate calibration is it possible to obtain a series of endogenous and unpredictable switches from REE to high BSE and viceversa. Several features contribute to the aim. Firstly correlation coefficient are $\gamma = 0.21$ (that makes $\beta^* = 1.01$ very near unity) and $\rho_v = 0.4$. For such values low BSE values are about half of high BSE ones, that in turn result to be relatively quite small. Therefore REE and high BSE basins of attraction have almost the same extent. Moreover we chose a bigger gain, namely $g_d = g_f = \varrho^{-1}/48$ in order to make estimates dynamics more volatile and hence jumps more likely.

Numerical simulation shows that dynamics similar to the latter can be generated considering more than two institutional forecasters with less extreme calibration[5]. Analytical results for such cases require a quite cumbersome computational analysis that is far beyond the scope of this work and will be object of future investigation.

[5] The programm is available upon request to the author.

6 Conclusion

Adaptive learning in macroeconomics has been always presented as a bounded rationality approach since a central hypothesis is that agents don't recognize the self-referential nature of the model. In other words, agents focus only on exogenous determinants of the economy by-passing all issues linked to interactions among them. This feature results as an ad-hoc departure from full rationality paradigm and, as such, it weakens the theoretic robustness of this approach. More importantly the bounded rationality hypothesis prevents the explicit modelling of interdependence between agents' expectations that is widely recognized to be responsible for crisis triggering. Gaballo (2009) shows how to extend the approach to deal with such issues. Here we have used such results to model endogenous changes in volatility regimes due to emergence of interdependence among agents' expectations. We have also shown a simple way to reconcile the standard use of adaptive learning approach with the idea agents recognize the self-referential nature of the economy.

We have investigated a simple Lucas-type monetary model in which inflation depends on expectation of current inflation and other exogenous determinants. In this setting we assumed expectations are interdependent in two respects. Firstly private sector evenly relies on two institutional forecasters. The latter are the only ones among agents having resource to gather and efficiently analyse information. In fact, each institutional forecaster implements statistical techniques to learn in real time the rational expectation, that is the fundamental inflation. The second way by which expectations are interdependent is due to behavioral uncertainty hypothesis. Behavioral uncertainty means that each institutional forecaster does not have perfect information about the other one's simultaneous expectations, but only a noisy signal of it. Given non-negligibility of institutional forecasters' expectations, they have incentive to condition their expectations to these noisy signals in order to minimize their forecast error variance. In particular, they have to assess whether or not actual deviations from the esteemed fundamental rate are due to idiosyncratic departure of others rationality from the rational one. In sum, institutional forecasters have to learn not only about the fundamental inflation rate (as in standard adaptive learning literature) but also about rationality of others.

We have proved how the interaction of these two channels of expectations interdependence and constant gain adaptive learning can give rise to two type of learneable equilibria, namely the REE and BSEs. The former occurs whenever both agents' estimates of the optimal weight of noisy behavioral observations converge in distribution to zero, the latter arises otherwise. BSE are equilibria for which volatility of behavioral noisy observations enters in the actual law of motion generating excess inflation volatility. More importantly, constant gain learning generates endogenous, unpredictable and persistent switches in volatility regimes. These changes are obtained without any aggregate shock exogenously imposed. On the contrary excess volatility is triggered by noises justified by behavioral uncertainty at a micro level. The model has the advantage of being perfectly consistent with REE behavior and, nevertheless, it has the potentiality to exhibit endogenous structural changes.

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