DSGE Model Nonlinearities

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Papers and software available at **https://web.sas.upenn.edu/schorf/**

DSGE Model Nonlinearities

Large body of recent work on DSGE model nonlinearities:

- stochastic volatility;
- effective lower bound on nominal interest rates;
- occasionally-binding financial constraints;
- general nonlinear dynamics in macro-financial models;
- (...)

Three Tasks

1. Model Solution
2. Model Estimation
3. Model Assessment

I will provide an overview of some of my recent collaborative research in these areas.

Task 1 – Model Solution

Nonlinear Model Solution

Reference: B. Aruoba, P. Cuba-Borda, and F. Schorfheide (2017): "Macroeconomic Dynamics Near the ZLB: A Tale of Two Countries," *Review of Economic Studies*, forthcoming.

- Perturbation solutions capture some nonlinearities but not all
  $\longrightarrow$ not well suited for occasionally-binding constraints.

- Example: ZLB/ELB for nominal interest rates

$$R_t = \max\left\{1,\ R_t^* e^{\epsilon_{R,t}}\right\}, \quad R_t^* = \left[r\pi_* \left(\frac{\pi_t}{\pi_*}\right)^{\psi_1} \left(\frac{Y_t}{Y_t^*}\right)^{\psi_2} \right]^{1-\rho_R} R_{t-1}^{\rho_R}.$$

- Two Challenges:
  1. capture "kinks" in decision rules;
  2. solution needs to be accurate in region of state-space that is relevant according to model AND according to data.

- Other issue in paper: multiplicity of equilibria, sunspots ...

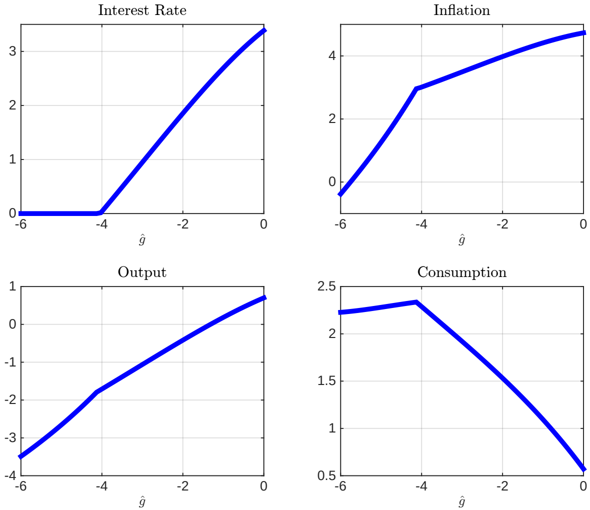
Challenge 1 – Kinks

- Consider decision rule $\pi(\mathcal{S}_t)$, states $\mathcal{S}_t = (R_{t-1}, y^*_{t-1}, d_t, g_t, z_t, \epsilon_{R,t})$
- "Stitch" two functions for each decision rule (endogenous "seam"):

$$\pi(\mathcal{S}_t;\Theta) = \begin{cases} f^1_\pi(\mathcal{S}_t;\Theta) & \text{if } R(\mathcal{S}_t) > 1 \\ f^2_\pi(\mathcal{S}_t;\Theta) & \text{if } R(\mathcal{S}_t) = 1 \end{cases}$$

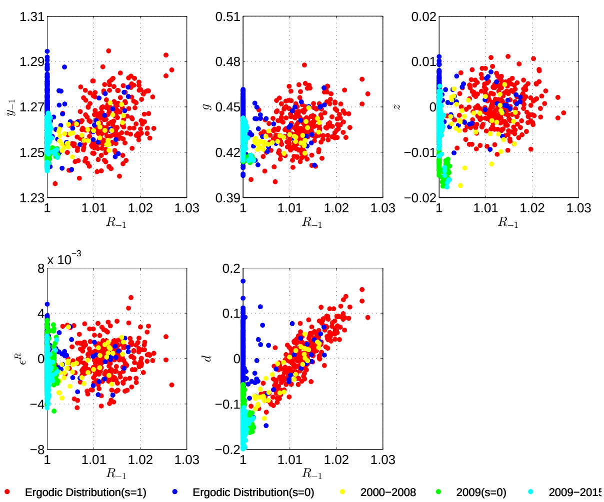
- f^i_j are linear combinations of a complete set Chebyshev polynomials up to 4^{th} order, with weights Θ.

Sample Decision Rules - Small-Scale NK Model for U.S.



Challenge 2 – Accuracy Where it Matters

Choose Θ to minimize sum squared residuals from the (intertemporal) equilibrium conditions over particular grid of points in state space



Task 2 – Model Estimation

Model Estimation – A Plug for Bayesian Inference...

$$p(\theta|Y) = \frac{p(Y|\theta)p(\theta)}{\int p(Y|\theta)p(\theta)d\theta}$$

- Treat uncertainty with respect to shocks, latent states, parameters, and model specifications uncertainty symmetrically.
- Condition inference on what you know (the data Y) instead of what you don't know (the parameter θ).
- Make optimal decision conditional on observed data.
- Large set of computational tools available.

Model Estimation

- Bayesian inference is implemented by sampling draws θ^i from the posterior $p(\theta|Y)$.
- Posterior samplers require evaluation of likelihood function: $\theta \longrightarrow$ model solution $\longrightarrow$ state-space representation $\longrightarrow p(Y|\theta)$.
- State-space representation $\longrightarrow p(Y,S|\theta)$:

$$
\begin{aligned}
y_t &= \Psi(s_t, t; \theta) + u_t, \quad u_t \sim F_u(\cdot;\theta) \\
s_t &= \Phi(s_{t-1}, \epsilon_t; \theta), \quad \epsilon_t \sim F_\epsilon(\cdot;\theta).
\end{aligned}
$$

- In order to obtain $p(Y|\theta) = \prod_{t=1}^T p(y_t|Y_{1:t-1}, \theta)$ we need to integrate out latent states S from $p(Y,S|\theta)$ $\longrightarrow$ use filter:
  - Initialization: $p(s_{t-1}|Y_{1:t-1}, \theta)$
  - Forecasting: $p(s_t|Y_{1:t-1}, \theta)$, $p(y_t|Y_{t-1})$
  - Updating: $p(s_t|y_t, Y_{1:t-1}) = p(s_t|Y_{1:t})$.

Particle Filtering

- Particle Filtering: represent $p(s_{t-1}|Y_{1:t-1})$ by $\{s^j_{t-1}, W^j_{t-1}\}_{j=1}^M$ such that

$$\frac{1}{M}\sum_{j=1}^{M} h(s^j_{t-1})W^j_{t-1} \approx \int h(s_{t-1})p(s_{t-1}|Y_{1:t-1})ds_{t-1}.$$

- Example: Bootstrap particle filter
  - Mutation/Forecasting: turn s^j_{t-1} into $\tilde{s}^j_t$: sample $\tilde{s}^j_t \sim p(s_t|s^j_{t-1})$.
  - Correction/Updating: change particle weights to: $\tilde{W}^j_t \propto p(y_t|\tilde{s}^j_t)W^j_{t-1}$.
  - Selection (Optional): Resample to turn $\{\tilde{s}^j_t, \tilde{W}^j_t\}_{j=1}^M$ into $\{s^j_t, W^j_t = 1\}_{j=1}^M$.
- Problem: naive forward simulation of Bootstrap PF leads to uneven particle weights $\longrightarrow$ inaccurate likelihood approximation!

Tempered Particle Filter

Reference: E. Herbst and F. Schorfheide (2017): "Tempered Particle Filtering," *NBER Working Paper*, 23448.

- Construct a sequence "bridge distributions" with inflated measurement errors. Define

$$p_n(y_t|s_t,\theta) \propto \phi_n^{d/2}|\Sigma_u(\theta)|^{-1/2}\exp\bigg\{ -\frac{1}{2}(y_t-\Psi(s_t,t;\theta))' \times\phi_n\Sigma_u^{-1}(\theta)(y_t-\Psi(s_t,t;\theta))\bigg\}, \quad \phi_1<\phi_2<\ldots<\phi_{N_\phi}=1.$$

- Bridge posteriors given s_{t-1}:

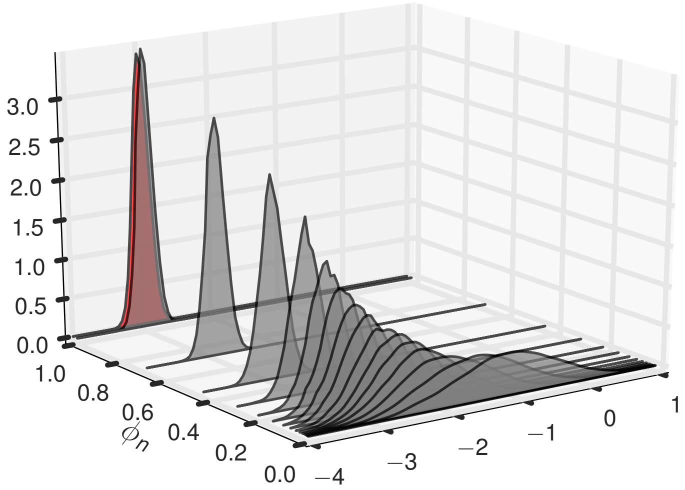
$$p_n(s_t|y_t,s_{t-1},\theta) \propto p_n(y_t|s_t,\theta)p(s_t|s_{t-1},\theta).$$

- Bridge posteriors given $Y_{1:t-1}$:

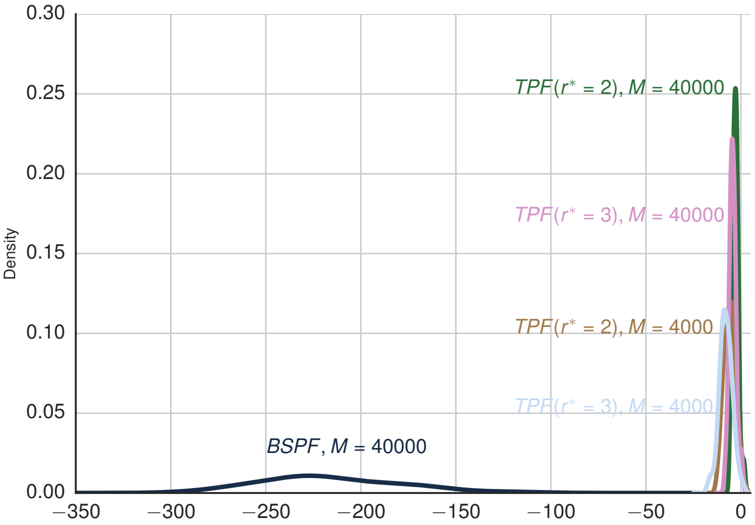
$$p_n(s_t|Y_{1:t}) = \int p_n(s_t|y_t,s_{t-1},\theta)p(s_{t-1}|Y_{1:t-1})ds_{t-1}.$$

- Traverse these bridge distributions with "static" Sequential Monte Carlo method (Chopin, 2002). References in stats lit: Godsill and Clapp (2001), Johansen (2016)

Bridge Posteriors: $p_n(s_t|Y_{1:t}),\ n = 1, \ldots, N_\phi$



Distribution of Log-lh Approx Error – Great Recession Sample



Putting it All Together

- Once a reasonably accurate likelihood approximation has been obtained, it can be embedded in a posterior sampler.
- The Full Monty is a real pain: see Gust, C., E. Herbst, D. Lopez-Salido, and M. E. Smith (2017): "The Empirical Implications of the Interest-Rate Lower Bound," *American Economic Review*, forthcoming.
- Potential shortcuts:
  - less accurate model solution;
  - cruder state extraction / likelihood approximation;
  - non-likelihood-based parameterization of model.
- Schorfheide, Song, Yaron (2017): slight short-cut in model solution $\longrightarrow$ conditionally-linear state-space representation $\longrightarrow$ efficient particle filter approximation of likelihood $\longrightarrow$ full Bayesian estimation.

Part III – Model Assessment

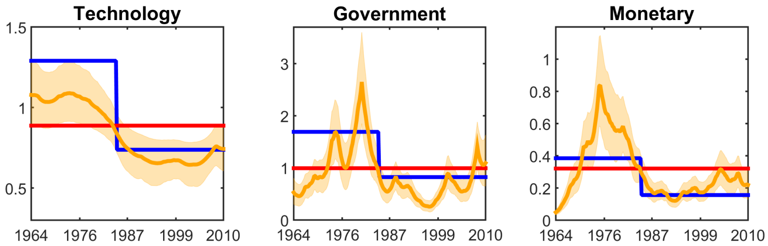
Can the nonlinearities in DSGE models correctly reproduce the nonlinearities in the data?

- Model Assessment (see Fernandez-Villaverde, Rubio-Ramirez, and Schorfheide (2016): "Solution and Estimation of DSGE Models," *Handbook of Macroeconomics*, Vol 2., Elsevier):
  - Relative fit: comparison with other models.
  - Absolute fit: can the DSGE reproduce salient features of the data? Violation of over-identifying restrictions?
- Linear VARs have been useful benchmark / reference model for the evaluation of linearized DSGE models:
  - testing over-identifying restrictions; model odds
  - comparison of model-implied autocovariances, spectra, impulse responses
  - (...)
- No obvious benchmark for evaluation of nonlinear models: generalized autoregressive models? bilinear models? ARCH-M? LARCH? regime-switching models? time-varying coefficient models? threshold autoregressions? smooth transition autoregressions?

Assessing Nonlinearities Can Be Delicate... An Example

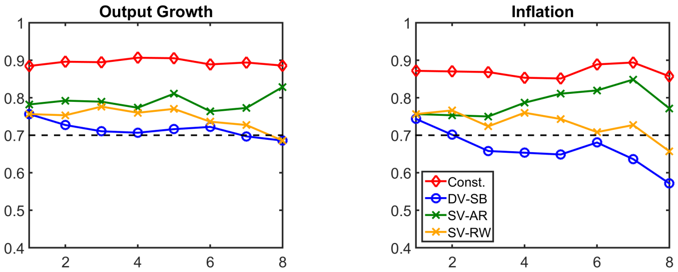
- Many papers argue that it is important to incorporate stochastic volatility in DSGE models.
- Important nonlinearity or device to capture rare events like Great Moderation, Great Recession?
- **Diebold, Schorfheide, Shin (2016):** Evaluate forecast performance of DSGE model with stochastic volatility versus structural break in volatility

Posterior Mean Structural Shock Volatilities / Final Data Vintage

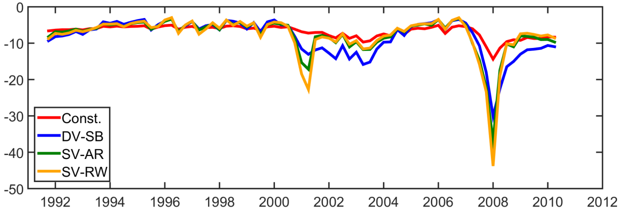


Assessing Nonlinearities Can Be Delicate... An Example: DSS (2016)

Coverage Rates of 70% Interval Forecasts, $h = 1, \ldots, 8$



Log Predictive Scores, $h = 4$



Assessing a DSGE Model With Smooth Nonlinearities

Reference: B. Aruoba, L. Bocola, and F. Schorfheide (2017): "Assessing DSGE Model Nonlinearities," *Manuscript*.

- Small-scale DSGE model with nominal price and wage rigidities.
- Price and wage adjustment costs are potentially asymmetric to capture downward rigidity, see Kim and Ruge-Murcia (JME, 2009):

$$\Phi(x) = \varphi\left(\frac{\exp(-\psi(x-x_*)) + \psi(x-x_*) - 1}{\psi^2}\right).$$

- Model consists of
  - households
  - intermediate goods producers
  - final goods producers
  - central bank / fiscal authority

DSGE Model Estimation

Some evidence for downward nominal rigidities:

	1960:Q1-2007:Q4		1984:Q1-2007:Q4	
Parameter	Mean	90% Interval	Mean	90% Interval
Price Rigidity				
PC Slope $\kappa(\varphi_p)$	0.02	[0.01, 0.04]	0.21	[0.12, 0.35]
Price Asymmetry ψ_p	150	[130, 175]	165	[130, 192]
Wage Rigidity				
Wage Adj Costs φ_w	18.7	[8.47, 38.1]	11.7	[5.34, 20.2]
Wage Asymmetry ψ_w	67.4	[33.2, 99.5]	59.4	[21.7, 90.9]

$$\Phi(x) = \varphi \left(\frac{\exp(-\psi(x - x_*)) + \psi(x - x_*) - 1}{\psi^2} \right).$$

Review: Perturbation Solution

- State $(s_{i,t})$ and control $(c_{i,t})$ variables evolve according to

$$\begin{aligned} c_{i,t} &= \psi_{1i}(\theta) + \psi_{2ij}(\theta)s_{j,t} + \psi_{3ijk}(\theta)s_{j,t}s_{k,t} \\ s^{end}_{i,t+1} &= \zeta^{end}_{1i}(\theta) + \zeta^{end}_{2ij}(\theta)s_{j,t} + \zeta^{end}_{3ijk}(\theta)s_{j,t}s_{k,t} \\ s^{exo}_{i,t+1} &= \zeta^{exo}_{2i}(\theta)s^{exo}_{i,t} + \zeta^{exo}_{3i}(\theta)\epsilon_{i,t+1}. \end{aligned}$$

- Perturbation solutions are easy to compute (DYNARE), improve accuracy near steady state (though not necessarily globally).

- BUT: are $\psi_{3ijk}(\theta)s_{j,t}s_{k,t}$ and $\zeta^{end}_{3ijk}(\theta)s_{j,t}s_{k,t}$ consistent with data?

Our Approach – Posterior Predictive Checks

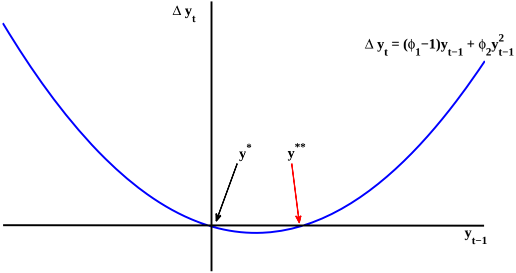
- Develop a nonlinear time series model that mimics structure of DSGE solution.
- Compare estimates of this model based on actual data and DSGE model-generated data.
- Alternative approaches:
  - Barnichon and Matthes (2016): create nonlinear benchmark or DSGE evaluation using Gaussian mixture approximation of moving ave. representation.
  - Ruge-Murcia (2016): indirect inference based on a VAR with higher-order terms and some DSGE model-implied zero restrictions.
  - Time-variation as / versus nonlinearity: literature on TVP VARs building on Cogley and Sargent (2002, 2005) and Primiceri (2005); evidence from time-varying model weights as in Del Negro, Hasegawa, and Schorfheide (2016).

A "Naive" Quadratic Autoregressive Model

- Generalized autoregressive (GAR) models (e.g. Mittnik, 1990): add quadratic terms to a standard autoregressive:

$$y_t = \phi_0 + \phi_1(y_{t-1} - \phi_0) + \phi_2(y_{t-1} - \phi_0)^2 + \sigma u_t$$

- Unattractive features: (i) multiple steady states; (ii) explosive dynamics.



- Problem is well-known in DSGE model solution literature $\longrightarrow$ pruning, e.g., Kim, Kim, Schaumburg, and Sims (2008), Lombardo (2010), Andreasen, Fernandez-Villaverde, and Rubio-Ramirez (2016) (...).

QAR(1,1): Specification

- We set $f_{uu} = 0$ to maintain a conditional Gaussian distribution and consider the system as a nonlinear state-space model:

$$
\begin{aligned}
y_t &= \phi_0 + \phi_1(y_{t-1} - \phi_0) + \phi_2 s_{t-1}^2 + (1 + \gamma s_{t-1})\sigma u_t \\
s_t &= \phi_1 s_{t-1} + \sigma u_t \qquad\qquad u_t \overset{\text{i.i.d.}}{\sim} \mathcal{N}(0,1)
\end{aligned}
$$

- Important properties:
  - Conditional linear structure facilitates calculation of moments; see Andreasen et al.
  - Stationary if $|\phi_1| < 1$.
  - Nonlinear impulse responses and conditional heteroskedasticity.

Relationship Between Nonlinear Dynamics and TVP Models

- Write nonlinear model as

$$y_t = f(y_{t-1}) + \sigma u_t = \underbrace{\phi(y_{t-1})}_{\phi_t} y_{t-1} + \sigma u_t.$$

- Could treat the estimation of $\phi(y_{t-1})$ nonparametrically, e.g., with prior

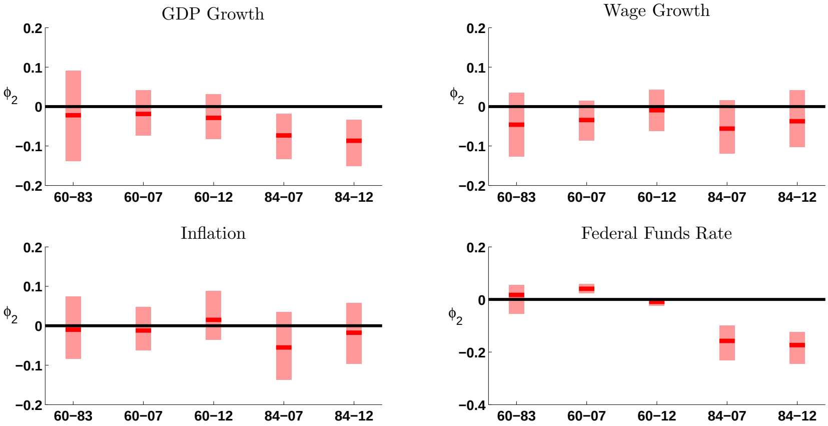
$$\phi(0) \sim N(\rho, \lambda), \quad \phi(y) - \phi(0) \sim N\big(0, \delta|y|\big).$$

- This is *ex ante* different from assuming that

$$y_t = \phi_t y_{t-1} + \sigma u_t, \quad \phi_t = \phi_{t-1} + \sigma_\eta \eta_t.$$

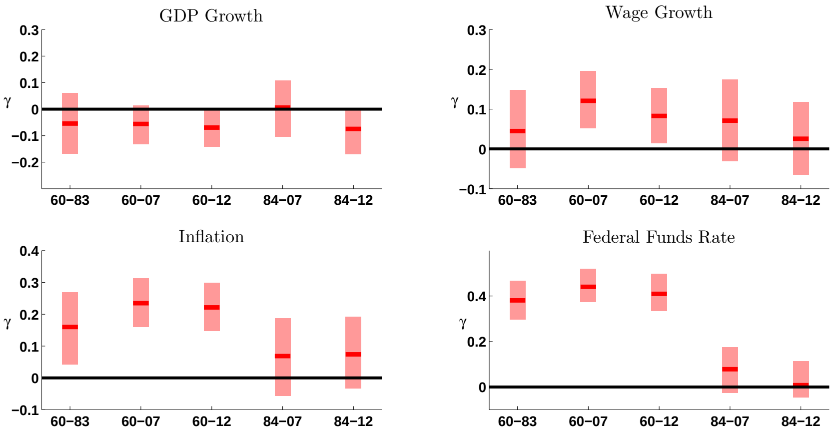
- State (y_t) dependence versus time dependence of ϕ_t.

Estimation of QAR(1,1) Model on U.S. Data – Φ_2



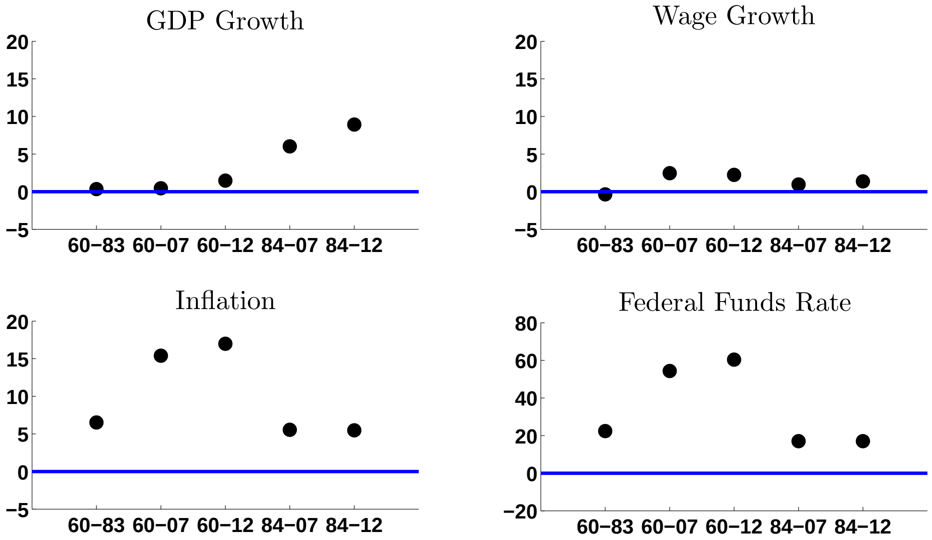
$$y_t = \phi_0 + \phi_1(y_{t-1} - \phi_0) + \phi_2 s_{t-1}^2 + (1 + \gamma s_{t-1})\sigma u_t, \qquad s_t = \phi_1 s_{t-1} + \sigma u_t$$

Estimation of QAR(1,1) Model on U.S. Data – γ

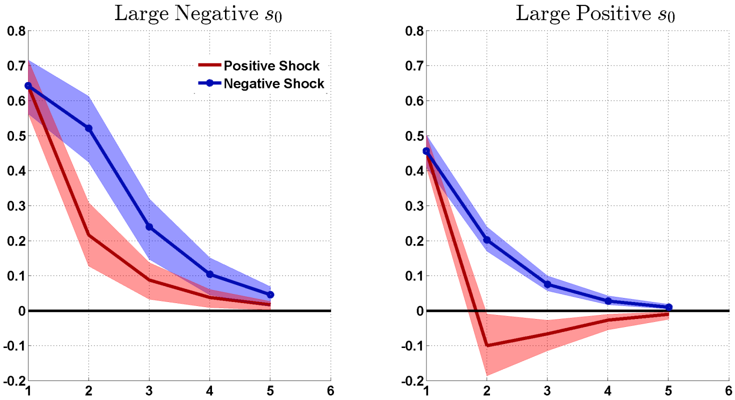


$$y_t = \phi_0 + \phi_1(y_{t-1} - \phi_0) + \phi_2 s_{t-1}^2 + (1 + \gamma s_{t-1})\sigma u_t, \qquad s_t = \phi_1 s_{t-1} + \sigma u_t$$

Log Marginal Data Density Differentials: QAR(1,1) versus AR(1)



Some Properties: Impulse Responses of GDP Growth (in Absolute Terms), 1984-2012 Sample

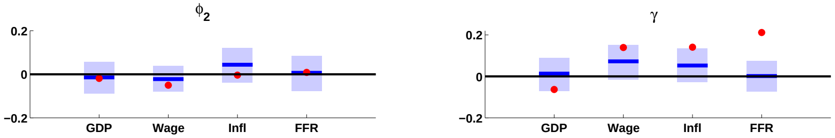


$$y_t = 0.53 + 0.36(y_{t-1} - \phi_0) - 0.09 s_{t-1}^2 + (1 - 0.07 s_{t-1}) 0.28 u_t$$

$$s_t = 0.36 s_{t-1} + 0.28 u_t$$

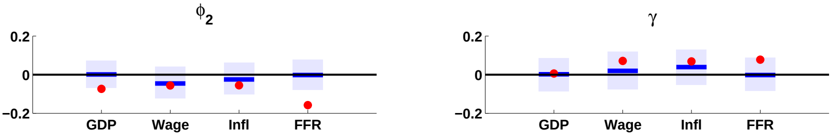
Posterior Predictive Checks

1960-2007 Sample



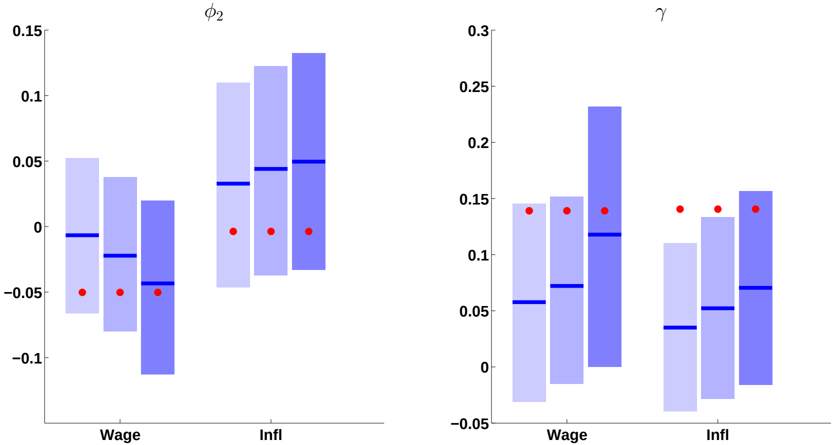
- QAR estimates from actual and model-generated data are similar.
- Only interest rates exhibit noticeable differences.
- Except for wage and inflation $\hat{\gamma}$, nonlinearities are generally weak.

1984-2007 Sample



- Model does not generate nonlinearity ($\hat{\phi}_2$) in GDP dynamics.

Effect of Adjustment Costs on Nonlinearities: 1960-2007 Sample



No asymmetric costs is $\psi_p = \psi_w = 0$ (light blue); high asymmetric costs is $\psi_p = \psi_w = 300$ (dark blue). Large red dots correspond to posterior median estimates based on U.S. data.

Summary of Empirical Results

- Some nonlinearities in U.S. data:
  - Post 1983: output growth displays sharp declines and slow recoveries.
  - 1960-2007: inflation and nominal wage growth display conditional heteroskedasticity.
  - Post 1983: downward adjustments in FFR seem to be easier than upward adjustments.
- DSGE model captures some but not all nonlinearities:
  - Conditional heteroskedasticity in inflation and nominal wage growth through asymmetric adjustment costs that penalize downward movements.
  - But no nonlinearities in model-implied output growth and FFR.

Conclusions

- Literature on methods and applications for DSGE models is well and alive!
- Significant progress in area of model solution and estimation techniques.
- More work needed on the model assessment:
  - Do nonlinearities in one area of model correctly propagate to other areas?
  - Does model perform well in crisis times?
  - Are nonlinearities strong enough so that they are measurable in “short” samples?