Limitations of demand constraints in stabilising financial markets with heterogeneous beliefs.

Daan in 't Veld[a]

[a] *CeNDEF, Department of Quantitative Economics, University of Amsterdam, Valckeniersstraat 65-67, 1018 XE Amsterdam, the Netherlands*

Preliminary: please do not quote without prior consultation

Abstract

This paper investigates the impact of constraining investment positions on financial market stability. Investors' demand is modelled in a well-known asset pricing model with heterogeneous beliefs. In particular, I generalise the heterogeneous agents model of Anufriev and Tuinstra (2013) to allow for leverage as well as short-selling constraints. I consider two examples of adaptive belief systems describing the coevolution of prices and investors' beliefs. First, if the market is inhabited by fundamentalist and chartist traders, demand constraints have the dangerous potential to restrict the stabilising fundamentalist strategy and can increase mispricing and price volatility. Second, if the market is inhabited by fundamentalists, optimists and pessimists with fixed beliefs, demand constraints drive down price volatility, but mispricing remains. The results suggest that there are considerable limitations of demand constraints in stabilising financial markets. Tight constraints can reduce the destabilising effect of investors with non-fundamental trading strategies, but any arising steady state price lies above the fundamental value.

1. Introduction

Should policy makers intervene in financial markets, and if so, how effective are certain regulations? During times of large financial distress, the possibly contributing role of excessive investment positions to asset price bubbles and crashes is heavily debated. It has been noted that leverage ratios increased to enormous amounts, which allowed investors to buy assets with money they did not actually have. Short-selling is an inverse phenomenon, where investors sell assets they do not actually own. This paper critically assesses the conjecture that constraining leverage and short-selling stabilises financial markets and restores

*D.intVeld@uva.nl

prices to fundamentals.

The main message of this paper is that the stabilising role of demand constraints is severely limited if investors have heterogeneous beliefs about future prices. This result is shown to be robust for various forms of belief formation by investors. The intuition for these limitations follows logically from the idea that different beliefs themselves can be stabilising or destabilising. As demand constraints are imposed on all market participants, there is no guarantee that destabilising beliefs become most restricted. In fact, this paper shows examples where demand constraints have adverse effects because the investors that should stabilise the market become restricted. It is possible that demand constraints lead to a new steady state, but then the market is dominated by the most optimistic investors and the steady state price lies always above the fundamental value.

To investigate leverage and short-selling constraints I use the well-known Brock and Hommes (1998) dynamic asset pricing framework with heterogeneous beliefs. The dynamical system arising in this model is called an *adaptive belief system* (ABS) and describes how prices and beliefs coevolve over time. Agents switch between beliefs if they observe that one of them has proven more profitable in the past. If agents observe that other beliefs are too biased and not profitable, a consensual belief consistent with a representative agent can arise endogenously. An attractive feature of adaptive belief systems is that strategies producing consistent losses are driven out of the market, which reduces the degrees of freedom in modelling heterogeneous beliefs.

If investors take large positions in financial markets, inevitably they should be different in some way: investors are only able to buy large amounts of stock if other are willing to sell these amounts. Heterogeneity is thus a key feature to model leverage and short-selling in financial markets. A considerable literature of bubbles and crises in asset markets with heterogeneous investors exists and is surveyed in e.g. Brunnermeier and Oehmke (2012). The research that uses heterogeneity to evaluate leverage and short-selling constraints is reviewed below.

Leverage has mainly been analysed in equilibrium models with heterogeneous, yet fully rational agents. Gromb and Vayanos (2002) proposes a model with financially constrained arbitrageurs who can only invest one out of two risky assets. Longstaff (2008) considers two types of agents who differ in risk aversion. In a more recent stream of the literature, Geanakoplos (2010) investigates the leverage cycle arising in a model with a continuum of investors who differ in their optimism, and shows how the margin changes over time. Simsek (2013) focuses on two belief types, optimists and pessimists, and models in greater detail the loan contract optimist use to leverage their investment. However, these papers evaluate how leverage affects equilibrium prices under rational agents with different characteristics, but say little about dynamic stability as I do in this paper.

The paper that comes closest to mine in investigating the dynamic effects of leverage is Thurner et al. (2012). They develop a quantitative dynamics model where fundamentalist investors buy an underpriced asset. Underpricing occurs when noise traders have driven the price below the fundamental value. The fundamentalists' optimal investment is proportional to their wealth[1], but leverage constraints bind the maximum amount they can invest. The results of the model show the downward spiral of margin calls: the natural buyers of the asset can turn into sellers when a price fall leads to a margin call. However, short-selling by investors is excluded, and therefore the model of Thurner et al. (2012) can only be applied to situations where the asset is underpriced. This paper combines possible effects of short-selling and leverage, and applies to both underpriced as overpriced assets.

Short-selling constraints have mostly been assessed as a policy tool on its own account and without considerations of leverage. In the seminal paper of Harrison and Kreps (1978), risk-neutral investors with different valuations about the future asset price would take infinitely large positions in the absence of short-selling constraints. With a full ban on short-selling, speculative bubbles arise: the price can exceed even the expectation of the most optimistic agent because this agent believes he can resell the asset at an even higher price in the future. Anufriev and Tuinstra (2013) evaluate the impact of short-selling constraints in a dynamic model with heterogeneous, boundedly rational agents. They show that trading costs for selling an asset short can increase mispricing and price volatility when the asset is overvalued.

To my knowledge, this is the first paper that explicitly combines the effects of leverage and short-selling constraints in a financial market model. Buss et al. (2013) compare different regulatory measures to control stock market volatility in a general equilibrium model with heterogeneous agents. Two of the regulations are short-selling constraints and leverage constraints, but these regulations are not imposed simultaneously in their model. Interestingly, although the methodology in Buss et al. (2013) is very different from the one followed in this paper, they arrive at the same conclusion that regulations can be effective in stabilising financial markets with heterogeneous beliefs.

This paper fits in a growing literature on using heterogeneous beliefs models for policy evaluations.[2] In particular, I generalise the model of Anufriev and Tuinstra (2013) to allow

[1] The wealth dynamics in the model of Thurner et al. (2012) are related to those of e.g. Chiarella and He (2001) or Anufriev and Bottazzi (2006).

[2] See also e.g. Westerhoff (2008).

for leverage constraints. Additional to the possibility of trading costs for short-selling, I introduce a maximum investment level that applies to all investors in every period. In this general model I evaluate the effects of leverage constraints in comparison with trading costs for short-selling, and I consider a mixed policy that restricts demand on both long and short positions. Overall, the analysis points out that the stabilising effects of demand constraints are considerably limited.

To check the robustness of these limitations, I evaluate the asset pricing model for different examples of adaptive belief systems. I consider not only a typical two-type model with fundamentalists versus chartists as in Anufriev and Tuinstra (2013), but also a three-type model with fixed beliefs of fundamentalists versus optimists and pessimists. These adaptive belief systems generate two typical stylised patterns observed in financial time series, namely endogenous bubbles and crashes, and oscillating behaviour around a fundamental price. By constraining investment positions along the fluctuations, the coevolution of prices and beliefs can be distorted.

The paper is organised as follows. Section 2 introduces the heterogeneous agents model with possible constraints on investors' demand. Section 3 presents the results for the aforementioned adaptive belief systems. In Section 4, I discuss the policy implications of the results for the stock market during the dot-com bubble and the financial crisis in 2008. The proofs can be found in the Appendix.

2. An asset pricing model with heterogenous beliefs and demand constraints

2.1. Individual asset demand

In this section I present an asset pricing model with heterogeneous agents and demand constraints, generalising Brock and Hommes (1998) and Anufriev and Tuinstra (2013). Denote y_t as the dividend payoff, p_t as the asset price and $z_{h,t}$ as the number of shares bought by investor type h, all at time t. Investors differ in their beliefs about what the price of the asset will be in the next time period: investor type h expects $E_{h,t}[p_{t+1}]$. Individual beliefs will influence demand $z_{h,t}$ and thereby the equilibrium price p_t. Apart from their beliefs, investors are homogeneous: they know that dividends are identically and independently distributed with mean $\bar{y}$, and believe the variance of asset prices is σ^2.[3]

It is assumed that investors are mean-variance maximisers as in Brock and Hommes

[3]Gaunersdorfer (2000) extends the model of Brock and Hommes (1998) with homogeneous but time-varying beliefs $\widehat{\sigma}_t^2$ about the variance of asset prices. For the model without demand constraints, the results are quite similar.

(1998):

$$\max_{z_{h,t}}(E_{h,t}[W_{h,t+1}] - \frac{a}{2}V_{h,t}[W_{h,t+1}]), \tag{1}$$

where a is the risk aversion, and $E_{h,t}[.]$ and $V_{h,t}[.]$ denote the beliefs of type h about the conditional expectation and variance. Wealth evolves as follows:

$$\begin{aligned} W_{h,t+1} &= RW_{h,t} + \rho_{t+1}z_{h,t} - R\tau(z_{h,t}) \\ &= RW_{h,t} + (p_{t+1} + \bar{y} - Rp_t)z_{h,t} - R\tau(z_{h,t}). \end{aligned} \tag{2}$$

The first term in equation (2) represents accumulation of wealth provided by a risk free asset under a fixed return $R \equiv 1 + r_f$, where r_f is the risk free rate. The second term denotes wealth effects from investing in the risky asset, depending on the number of shares $z_{h,t}$ and the excess return ρ_{t+1} of stocks over the risk free asset. This excess returns consists of the next period's price and dividend payoff less the opportunity costs of investing in the risky asset. The last term in the wealth equation (2) is $\tau(z_{h,t})$, the trading costs for selling assets short, and has been introduced in Anufriev and Tuinstra (2013) as follows:

$$\tau(z_{h,t}) = \begin{cases} 0 & \text{if } z_{h,t} \geq 0 \\ \\ T|z_{h,t}| & \text{if } z_{h,t} < 0, \end{cases} \tag{3}$$

implying that for every unit of the asset sold short the investor has to pay a fixed tax of T.

The additional assumption in this paper over the model of Anufriev and Tuinstra (2013) is that traders face leverage constraints. Given their own limited amount of wealth, they have to attract loans if they want to buy large amounts of assets. Denoting $\lambda^{max} \geq 1$ as the maximum leverage level, the leverage constraint can be written as:

$$\begin{aligned} p_t z_{h,t} &\leq \lambda^{max} W_{h,t} \\ &\leq I^{max}_{h,t}. \end{aligned} \tag{4}$$

The variable $I^{max}_{h,t}$ represents the *maximum investment level* agent type h can invest as a monetary amount.

In this paper I assume that leverage constraints are fixed, i.e. $I^{max}_{h,t} = I^{max} > 0$ for all h,t. In other words, while wealth accumulation is the main reason for investment, wealth *levels* are assumed to be constant over time and over all types. This means that investors have a fixed amount of working capital to invest in the financial market; profits $W_{h,t} - W_{h,t-1}$ are assumed to be are withdrawn at the end of period t to e.g. employers or shareholders. While a constant wealth level is clearly an extreme assumption, it provides a natural starting point for evaluating the effects of leverage constraints on asset price dynamics.[4]

In the absence of trading costs and if the maximum investment level is sufficiently high, the optimal demand follows from the unconstrained maximisation problem (1) and is a decreasing linear function in the price p_t:

$$\begin{aligned} z^*_{h,t}(p_t) &= \frac{E_{h,t}[\rho_{t+1}]}{aV_{h,t}[\rho_{t+1}]} \\ &= \frac{E_{h,t}[p_{t+1}] + \bar{y} - Rp_t}{a\sigma^2}. \end{aligned} \tag{5}$$

However, in general with possibly binding leverage and/or short-selling constraints, the linear relation between price and demand is severed.

First, consider the possibility that trader type h is leverage constrained as in (4) because the optimal investment $z^*_{h,t}$ is too high under price p_t. In this case, when the price p_t decreases further to 0, demand $z^*_{h,t}$ will become even higher. At some point though the acquisition costs $p_t z^*_{h,t}$ will start decreasing and the leverage constraint will stop being binding. In other words, demand is leverage constrained for a bounded range of strictly positive prices.[5] More precisely, the leverage constraint is binding when

$$z^*_{h,t} \geq I^{max}/p_t, \tag{6}$$

or equivalently

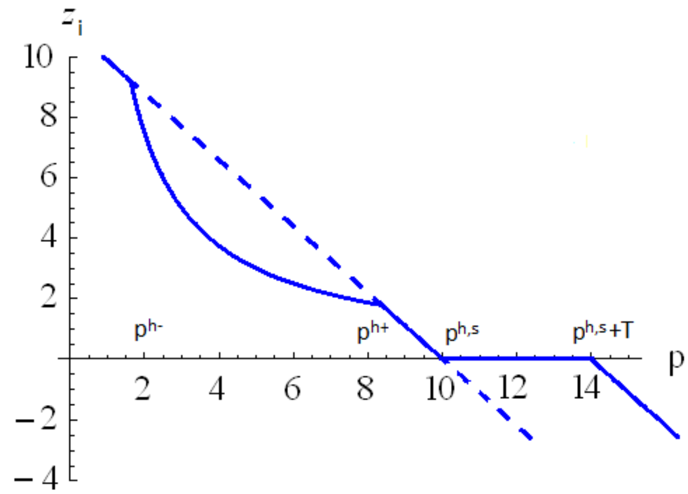
$$p_t \in [p_t^{h-}, p_t^{h+}],\ p_t^{h\pm} \equiv \frac{1}{2R}\left(E_{h,t}[p_{t+1}] + \bar{y} \pm \sqrt{(E_{h,t}[p_{t+1}] + \bar{y})^2 - 4a\sigma^2 I^{max} R}\right). \tag{7}$$

So there are two *cut-off values* $p_t^{h\pm}$ that define the area in which agent type h is leverage constrained. Of course, if the determinant is negative the cut-off values do not exists, and the agent is never constrained. In the constrained area the agent demands his maximal number of shares I^{max}/p_t. See Figure 1 for an example of an demand schedule where is seen to be the level of demand is lower in the area $[p^{h-}, p^{h+}]$.

[4] A more developed model with wealth dynamics could exhibit *feedback of leverage constraints*. When buyers become leverage constrained, prices drop, which decreases the buyers' profits and wealth accumulation. Less wealth make the buyers even more constrained, driving prices further down.

[5] It can be checked that $p_t z^*_{h,t} = 0$ for $p_t = 0$, such that demand can only be leverage constrained for strictly positive prices.

Figure 1: Demand z_h as a function of the price p (solid line) compared to linearly decreasing unconstrained demand (dashed line). For low prices $p^{h-} < p < p^{h+}$, demand is driven down by leverage constraints to I^{max}/p. For high prices $p^{h,s} < p < p^{h,s} + T$, demand is zero, while for even higher prices $p > p^{h,s} + T$ the short-selling tax T is covered and demand is again linearly decreasing.



Second, Anufriev and Tuinstra (2013) have observed that similar cut-off values exists for the short-selling constraint. The short-selling tax has to be paid when $z^*_{h,t} < 0$, or

$$p_t > p_t^{h,s} = \frac{1}{R}(E_{h,t}[p_{t+1}] + \bar{y}). \tag{8}$$

When the price p_t increases above $p_t^{h,s}$, the demand function stays flat as both buying and short-selling is not beneficial. For even higher prices, the expected excess return can cover the additional cost of T per unit sold short have to be paid. Both leverage and short-selling constraints generate nonlinear parts in the demand curve.

Taking the two constraints together gives a demand schedule that consists of five pieces:[6]

$$z_{h,t}(p_t) = \begin{cases} \frac{1}{a\sigma^2}(E_{h,t}[p_{t+1}] + \bar{y} - Rp_t) & \text{if } p_t < p_t^{h-} \\ I^{max}/p_t & \text{if } p_t \in [p_t^{h-}, p_t^{h+}] \\ \frac{1}{a\sigma^2}(E_{h,t}[p_{t+1}] + \bar{y} - Rp_t) & \text{if } p_t \in [p_t^{h+}, p_t^{h,s}] \\ 0 & \text{if } p_t \in [p_t^{h,s}, p_t^{h,s} + T] \\ \frac{1}{a\sigma^2}(E_{h,t}[p_{t+1}] + \bar{y} - R(p_t - T)) & \text{if } p_t > p_t^{h,s} + T \end{cases} \tag{9}$$

Figure 1 illustrates the nonlinear individual demand for trader type h as a function of p_t, for given values of $E_{h,t}[p_{t+1}]$ and parameters $\bar{y}$, R, $a\sigma^2$, T and I^{max}. It shows how the demand function is affected by the two main parameters of interest: T, the trading costs for short-selling, and I^{max}, the maximal investment level. For $T = 0$ and $I^{max} = \infty$, demand is unconstrained. The limit case $T = \infty$ represents a full ban on short-selling. A policy maker can either increase T, decrease I^{max} or both, in order to influence demand of individual investors and thereby the stability of the market.

2.2. Market equilibrium

Given the individual demand function and a population of agents, the question is which price brings the market in equilibrium. I consider H types of agents who differ in their beliefs $E_{h,t}[p_{t+1}]$ for $h \in 1, ..., H$. Denote the fractions of the different types at time t as $n_{h,t}$. The supply of the asset is fixed at z_s per trader. The intertemporal market equilibrium price p_t is implicitly defined by the market clearing equation:

$$\sum_{h=1}^{H} n_{h,t} z_{h,t}(p_t) = z_s \tag{10}$$

First consider the case that agents are rational. Then all investors can be described by a representative type ($H = 1$) with perfect foresight:

$$E_t^*[p_{t+1}] = p_{t+1}. \tag{11}$$

[6]One can check that $p_t^{h,s} > p_t^{h+}$, such that the interval $[p_t^{h+}, p_t^{h,s}]$ is non-empty.

In equilibrium this representative agent will hold z_s units of the asset, that is, if this investment level is allowed by the leverage constraint (4). In case the leverage constraint is not binding, the equilibrium price is

$$p_f = \frac{\bar{y} - a\sigma^2 z_s}{r_f}, \tag{12}$$

and denoted as the *fundamental price* p_f. In this paper I assume that the representative rational agent is not constrained by the maximum investment level:

Assumption 1. *Leverage constraints are sufficiently loose for the fundamental price (12) to hold, i.e.:*

$$\begin{aligned} I^{max} &> p_f z_s \\ &> \frac{\bar{y} - a\sigma^2 z_s}{r_f} z_s \end{aligned} \tag{13}$$

Now consider the case $H > 1$ with different beliefs $E_{h,t}[p_{t+1}]$. In the absence of trading costs and if the maximum investment level is sufficiently high, the equilibrium price can be derived from optimal unconstrained demand (5) and the market clearing equation (10):

$$p_t = \frac{1}{R}\left(\sum_{h=1}^{H} n_{h,t} E_{h,t}[p_{t+1}] + \bar{y} - a\sigma^2 z_s\right) \tag{14}$$

In general under demand constraints, it becomes unfeasible to derive the equilibrium price analytically, because the individual demand schedules are highly nonlinear. The following proposition assures that a unique solution exists:

Proposition 1. *Consider an asset market with heterogeneous beliefs and possible demand constraints. If $z_s > 0$, there exists a unique solution to $\sum_{h=1}^{H} n_{h,t} z_{h,t}(p_t) = z_s$ providing the market clearing price p_t at time t.*

Proof. See Appendix A □

The proof of Proposition 1 follows similar reasoning as in Anufriev and Tuinstra (2013). Individual demand is strictly decreasing when it is positive (see Figure 1), which ensures a unique solution for the market clearing price if outside supply z_s is strictly positive.[7]

[7] A prerequisite for the existence of a unique equilibrium is that leverage constraints are modeled as a

The next section treats the price dynamics in this model if agents can switch between strategies. Both the fraction $n_{h,t}$ and the beliefs $E_{h,t}[p_{t+1}]$ can change over time. Using the result of Proposition 1 that a unique price p_t exists in every period t, I analyse the effects of demand constraints on the dynamics.

3. Stylised adaptive belief systems

In this section I introduce the dynamics in the asset pricing market if agents can switch between different belief strategies. The fractions of agents with different types are updated using the multinomial logit model as in Brock and Hommes (1997):

$$n_{h,t+1} = \frac{e^{\beta U_{h,t}}}{\sum_{j=1}^{H} e^{\beta U_{j,t}}}. \tag{15}$$

The parameter β is the *intensity of choice* with which agents react on differences in performance. If $\beta = 0$, agents do not switch and fractions are evenly distributed over the H rules: $n_{h,t+1} = 1/H$ for all t. For increasing β, agents are more sensitive to performance in the last period and switch quicker to better performing rules.

For the performance $U_{h,t}$ of the different rules, the main component is formed by the excess returns ρ_t times the asset holdings $z_{h,t-1}$. Apart from excess returns, agents consider two possible sources of costs: first, the trading costs $R\tau(z_{h,t-1})$, depending on whether the asset is sold short, and second, exogenous costs C_h associated with gathering information to use strategy h. The performance measure is therefore given by:

$$U_{h,t} = \rho_t z_{h,t-1} - R\tau(z_{h,t-1}) - C_h. \tag{16}$$

In describing the expectation rule of the agents and price dynamics, I follow the convention to write prices in deviation from the fundamental p_f:

$$x_t = p_t - p_f. \tag{17}$$

In terms of deviations from the fundamental price, the excess return on stocks is:

$$\begin{aligned} \rho_{t+1} &= p_{t+1} + \bar{y} - Rp_t \\ &= x_{t+1} - Rx_t + a\sigma^2 z_s, \end{aligned} \tag{18}$$

maximum investment level I^{max} in a monetary amount. If instead a maximum were imposed on the amount of shares $z_{h,t}$, the demand would be a constant in p_t for two ranges of prices; then it would be possible that aggregate demand $\sum_{h=1}^{H} n_{h,t} z_{h,t}$ were constant for some range of prices.

and the demand schedule becomes:

$$z_{h,t}(x_t) = \begin{cases} \frac{1}{a\sigma^2}(E_{h,t}[x_{t+1}] - Rx_t) + z_s & \text{if } x_t < x_t^{h-} \\ I^{max}/(x_t + p_f) & \text{if } x_t \in [x_t^{h-}, x_t^{h+}] \\ \frac{1}{a\sigma^2}(E_{h,t}[x_{t+1}] - Rx_t) + z_s & \text{if } x_t \in [x_t^{h+}, x_t^{h,s}] \\ 0 & \text{if } x_t \in [x_t^{h,s}, x_t^{h,s} + T] \\ \frac{1}{a\sigma^2}(E_{h,t}[x_{t+1}] - R(x_t - T)) + z_s & \text{if } x_t > x_t^{h,s} + T \end{cases} \tag{19}$$

using appropriate definitions for $x_t^{h\pm}$ and $x_t^{h,s}$ corresponding with $p_t^{h\pm}$ and $p_t^{h,s}$.[8]

As Brock and Hommes (1998), I use simple beliefs that are linear in the last observation:

$$E_{h,t}[x_{t+1}] = b_h + g_h x_{t-1}. \tag{20}$$

As will become clear, this simple structure of belief formation is sufficient to capture a number of important cases. Asset pricing models inhabited by agents with beliefs of the form(20) are able to generate important properties in medium-run asset pricing dynamics, namely recurrent bubbles and crashes, and oscillations around a fundamental value.

I use two examples of adaptive belief systems (ABS). In Section 3.1, I follow Anufriev and Tuinstra (2013) in taking the ABS with fundamentalists versus chartists, where fundamentalists pay a cost for acquiring information about the fundamental value. In Section 3.2, I consider another example of an ABS from Brock and Hommes (1998) with three fixed beliefs ($g_h = 0\ \forall h$) and without information costs. Here the population of investors consists of fundamentalists, optimists and pessimists. In both examples, fundamental traders can*not* drive out irrational traders even for increasingly large intensity of switching.

[8] In contrast to Brock and Hommes (1998) and Anufriev and Tuinstra (2013), price dynamics are *not* independent on the value of the fundamental price. In the leverage constrained part in the demand function (9), I^{max}/p_t, the term p_f cannot be eliminated.

3.1. Fundamentalists versus chartists

Assume that there are two types of agents, the first *fundamentalists* and the second *chartists*:

$$
\begin{aligned}
E_{1,t}[x_{t+1}] &= 0, && (21)\\
E_{2,t}[x_{t+1}] &= b + g x_{t-1}. && (22)
\end{aligned}
$$

Fundamentalists pay a fixed cost $C > 0$ to find the fundamental value. Chartists do not acquire information to calculate the fundamental value, but update their beliefs given the last observed price x_{t-1} to extrapolate trends with some possible bias b. For simplification I set $b = 0$, such that chartists extrapolate trends around the fundamental value:[9]

$$E_{2,t}[x_{t+1}] = g x_{t-1}. \tag{23}$$

For the baseline parametrisation in the fundamentalist-chartist market (irrespective of demand constraints), I follow Anufriev and Tuinstra (2013):

Definition 1 (Baseline parametrisation under fundamentalists and chartists)**.** *Parameters:* $z_s = 0.1$*,* $a\sigma^2 = 1$*,* $r_f = 0.1$*,* $\bar{y} = 1.1$ *(*$\rightarrow p_f = 10$*),* $g = 1.2$*,* $C = 1$ *and* $\beta = 4$*.*

Figure 2 shows the benchmark dynamics without constraints. Figure 2a shows two attractors in the (x_{t-1}, x_t)-plane. In Figure 2b, the left panels show dynamics on the upper attractor; the right panels on the lower attractor. The three panels in vertical order depict the evolution of prices, fractions of fundamentalists and positions, respectively.

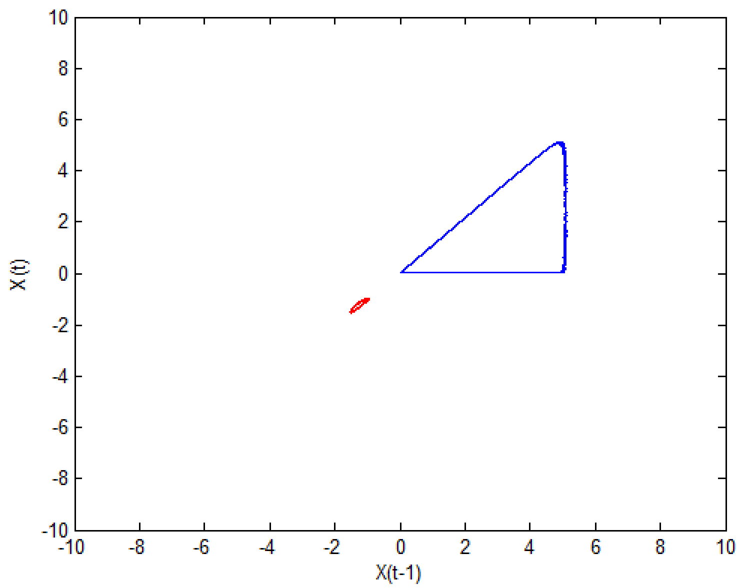
In this baseline model, the fundamental price is unstable for the relatively large extrapolation factor of chartists $g > R$. The reason is that around the steady state chartists earn equally large excess returns as fundamentalists, while circumventing the information costs. As chartists attract more agents, market prices are influenced by trend-following behaviour, and price bubbles arise endogenously. Gradually, though, as chartists overshoot the growing prices x_t more and more by their extrapolative expectations $g x_{t-1}$, fundamentalists regain followers and eventually there is a abrupt crash. For this choice of parameters, there are also small recurrent 'bubbles' or troughs below the fundamental value.

Recurrent bubbles are a robust phenomenon in this model, and it is possible to give

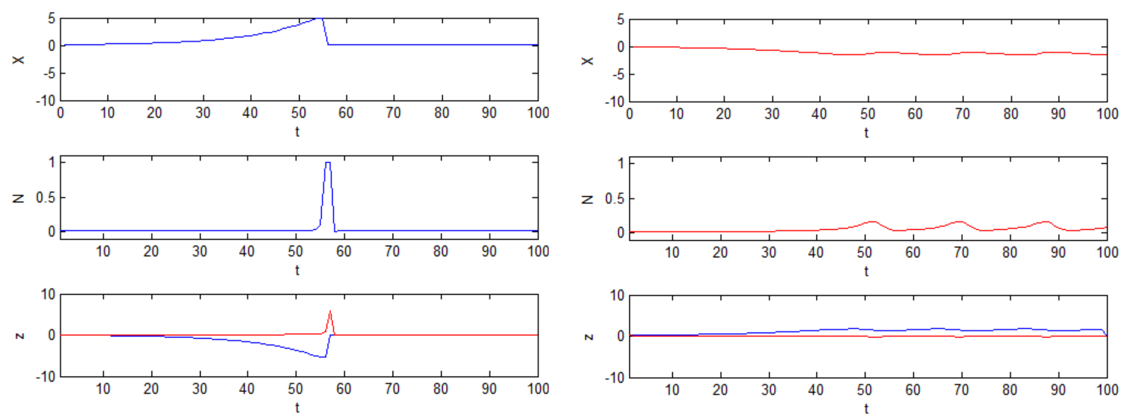
[9] One can argue that unbiased extrapolation requires p_f to be in the information set of the chartists, inconsistent with the information-gathering costs C of fundamentalists. However, for high values $g > R$ the term $g x_{t-1}$ generally dominates a relatively small bias b. For prices x_t close to zero, a larger bias of chartists is equivalent to a smaller C.

Figure 2: Phase plot (a) and time series (b) for the benchmark ABS with fundamentalists and chartists without constraints. The baseline parametrisation and $T = 0$, $I^{max} = \infty$.

(a) Phase plot showing two quasi-periodic attractors in different colors



(b) Time series. *Left panels:* dynamics on the upper attractor. *Right panels:* dynamics on the lower attractor. *Upper panels:* prices. *Middle panels:* fraction of fundamentalists. *Lower panels:* positions of fundamentalists (blue) and chartists (red).



bounds for how large bubbles can grow. Lemma 4 of Brock and Hommes (1998) states that for $z_s = 0$, $R < g < R^2$ and $\beta = \infty$ the dynamics are bounded. The following proposition specifies the bounds of price dynamics for $\beta = \infty$ given any value of $z_s > 0$.

Proposition 2. *Consider the ABS with fundamentalists and chartists without demand constraints. Assume $z_s > 0$, $C > 0$, and $\beta = \infty$. Then:*

1. *for $0 < g < R$, the fundamental steady state $E_1 = (x^*, n_1^*) = (0,0)$ is the unique, globally stable steady state.*

2. *for $R < g < R^2$, the fundamental steady state $E_1 = (0,0)$ is locally unstable. Price dynamics are bounded and converge to the (locally unstable) steady state $E_1 = (0,0)$. For an initial price above the fundamental $x_0 = \varepsilon > 0$, the dynamics are bounded by some smallest interval $[0, x^{max})$ with*

$$x^{max} \equiv \frac{g}{R} \frac{\sqrt{(a\sigma^2 z_s)^2 + 4(1-g/R^2)a\sigma^2 C} + a\sigma^2 z_s}{2(R^2/g-1)} \tag{24}$$

   For an initial price below the fundamental $x_0 = \varepsilon < 0$, the dynamics are bounded by some smallest interval $(x^{min}, 0]$ with

$$x^{min} \equiv -\frac{g}{R} \frac{\sqrt{(a\sigma^2 z_s)^2 + 4(1-g/R^2)a\sigma^2 C} - a\sigma^2 z_s}{2(R^2/g-1)} \tag{25}$$

3. *for $R^2 < g < R^2(1 + \frac{a\sigma^2 z_s^2}{4C})$, the fundamental steady state $E_1 = (0,0)$ is locally unstable. Price dynamics with an initial price above the fundamental $x_0 = \varepsilon > 0$ diverge. For an initial price below the fundamental $x_0 = \varepsilon < 0$, price dynamics converge to the (locally unstable) steady state $E_1 = (0,0)$ and are bounded by some smallest interval $(x^{min}, 0]$ with*

$$x^{min} \equiv -\frac{g}{R} \frac{\sqrt{(a\sigma^2 z_s)^2 - 4(g/R^2-1)a\sigma^2 C} - a\sigma^2 z_s}{2(1-R^2/g)} \tag{26}$$

4. *for $g > R^2(1 + \frac{a\sigma^2 z_s^2}{4C})$, the fundamental steady state $E_1 = (0,0)$ is locally unstable and all price dynamics with an initial price $x_0 = \varepsilon \neq 0$ diverge.*

Proof. See Appendix A □

Proposition 2 gives important insights about the magnitude of bubbles when the intensity of choice is large but finite. For $R < g < R^2$, prices move away from the fundamental with a rate close to g/R but remain within endogenous bounds. During the bubble chartists overshoot the evolution of prices and, with finite β, fundamentalists start to attract more followers before x_t reaches x^{max} or x^{min}. After the bubble is being stalled, prices return very closely to the unstable steady state, and a new bubble appears.

For the special case $z_s = 0$, prices are bounded symmetrically by

$$|x_t| < \frac{g^2}{R^2}\sqrt{\frac{a\sigma^2 C}{R^2 - g}}, \tag{27}$$

a value increasing in g, $a\sigma^2$ and C, and decreasing in R. As already found by Anufriev and Tuinstra (2013), when there is positive supply $z_s > 0$, bubbles are larger above the fundamental value than below. Proposition 2 makes clear that for $C \downarrow 0$, bubbles below the fundamental value disappear ($x^{min} = 0$) and the upperbound becomes:

$$x^{max} = \frac{g^2}{R}\frac{a\sigma^2 z_s}{R^2 - g}, \tag{28}$$

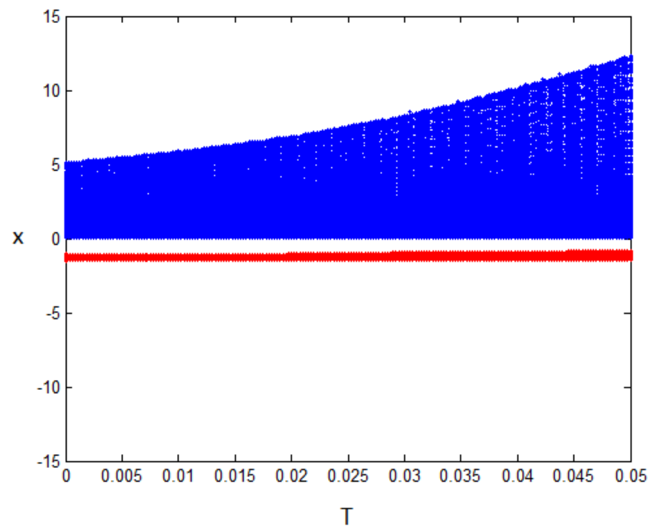
which is linearly increasing in z_s. The magnitude of bubbles depends intuitively on the parameters and can be evaluated analytically, which helps to investigate the effects of demand constraints in the next sections.

3.1.1. *Short-selling constraints in the two-type market*

First I consider short-selling constraints in this market. Figure 3 gives the bifurcation diagrams for the short-selling tax T for the baseline parametrisation, given $I^{max} = \infty$ to exclude the leverage constraint. The plot shows two attractors of time series given the parameters choices for different initial conditions. In the model with fundamentalists and chartists, prices can never cross the fundamental p_f and the time series x_t keeps the same sign as the initial value x_0. So in this model (without noise) prices are either always above the fundamental or always below the fundamental.

Above the fundamental value an increase in T increases average mispricing and volatility in the market. The reason is that as the bubble builds up above the fundamental value fundamentalists want to sell large amounts of the asset short: see the lower left panel of Figure 2b. Because the large number of short-sold shares carry high trading costs, the fundamentalist strategy is less profitable relative to the case without a short-selling tax. Chartists remain the dominant agent type and continue to 'ride' the bubble by extrapolating price for a longer time. As prices are driven up further, at some point fundamentalists' excess returns $\rho_t z_{1,t}$ of betting on a price drop will become so large to cover even the additional

Figure 3: Bifurcation diagram for short-selling tax T given the baseline parametrisation and $I^{max} = \infty$. Small noise is added to the price x_t ($\sigma = 10^{-3}$ with the same sign as x_t) to avoid that the numerical simulation gets stuck in the locally unstable steady state $x_t = 0$. Two attractors are indicated in different colors.



short-selling tax $R\tau(z_{h,t})$, and prices crash back close to the fundamental price.

On the other hand, below the fundamental value, the effects of a trading tax are negligible, as chartists short sell the assets but in relatively small amounts. This can be checked in the lower right panel of Figure 2. The bifurcation diagram for T in Figure 3 confirms the main result of Anufriev and Tuinstra (2013), namely that costs on short-selling increase mispricing and price volatility in the two-type market when the asset is overvalued.

3.1.2. Leverage constraints in the two-type market

To allow for an unconstrained fundamental value as in Assumption 1, the parameter choices in Definition 1 require $I^{max} \geq 1$. The effects of the leverage constraints on the dynamics are shown in Figure 4. On the lower attractor, bubbles become larger as the constraint is tightened, an effect that is similar for short-selling constraints on the upper attractor. The reason is that the fundamentalist strategy, which should drive back prices to the fundamental, requires much more extreme positions and soon face the leverage constraint (see Figure 2, lower right panel).

On the upper attractor, there are no visible effects for large values of I^{max}. The leverage constraint should restrict the price to rise far above the fundamental value, as chartists become constrained by the maximum investment level I^{max}. Because the chartists have relatively low demand levels along the bubble (see Figure 2, lower left panel), the leverage constraint only becomes binding after fundamentalists start have a higher performance and the crash already started.

However, in contrast to short-selling constraints, tight leverage constraints can have a stabilising effect, if prices are above the fundamental value. As seen in the upper part of Figure 4, leverage constraints do affect the dynamics for l^{max} smaller 3. In this case chartists become constrained during a rising market, making the bubble burst faster.

Figure 4: Bifurcation diagram for leverage constraint l^{max} given the baseline parametrisation and $T = 0$. Small noise is added to the price x_t ($\sigma = 10^{-3}$ with the same sign as x_t) to avoid that the numerical simulation gets stuck in the locally unstable steady state $x_t = 0$. Two attractors are indicated in different colors.

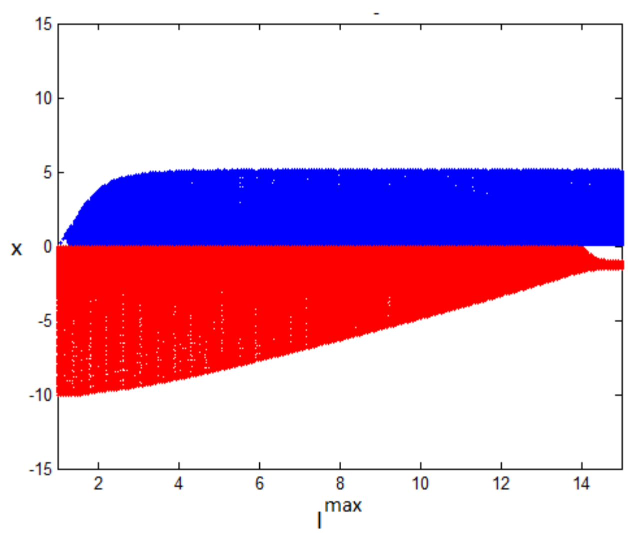
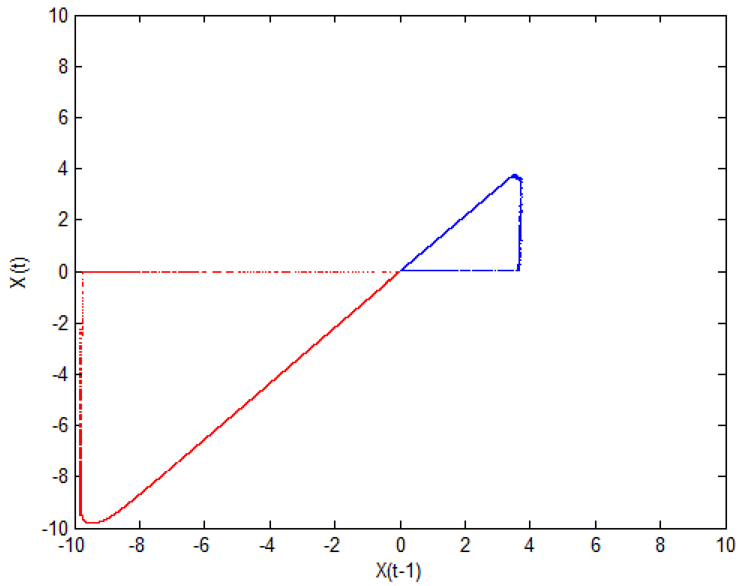


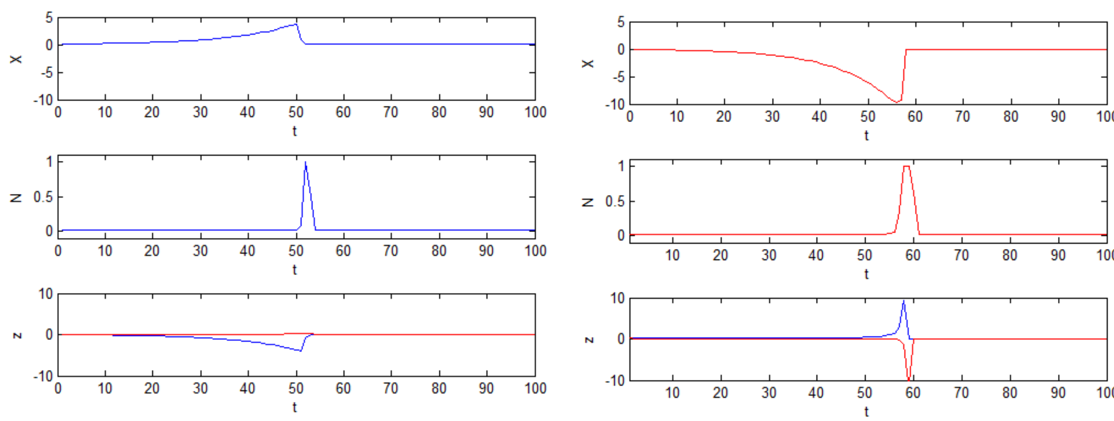
Figure 5 shows the phase plots and time series for a relatively tight leverage constraint $l^{max} = 2$ (without short-selling tax, $T = 0$). In Figure 5b, the evolution of prices, fractions and demand levels are presented on both attractors. On the upper attractor (left panels), chartists have small yet positive positions $z_{2,t}$ during the complete time series, through which the leverage constraint invokes a slightly earlier switch to the fundamentalist strategy and therefore a smaller bubble. On the lower attractor (right panels), fundamentalists can buy only small amounts leading to a large drop of the price x_t to almost -10.

Figure 5: Phase plot (a) and time series (b) for the benchmark ABS with fundamentalists and chartists. The baseline parametrisation and $T = 0$, $I^{max} = 2$.

(a) Phase plot showing two quasi-periodic attractors in different colors



(b) Time series. *Left panels:* dynamics on the upper attractor. *Right panels:* dynamics on the lower attractor. *Upper panels:* prices. *Middle panels:* fraction of fundamentalists. *Lower panels:* positions of fundamentalists (blue) and chartists (red).



3.1.3. Combining the two demand constraints in the two-type market

Taking the results in the two-type ABS together, a recurring point is that constraints have the dangerous potential to restrict the fundamentalist strategy, that requires larger positions. As fundamentalists are the stabilising type in the market, both constraints can increase price volatility and mispricing, but on different sides of the fundamental price. Short-selling constraints have destabilising effects above the fundamental price, when the fundamentalist strategy requires selling the asset; leverage constraints destabilise below the fundamental when it requires buying the asset. In both cases it takes a longer bubble for traders to switch to the fundamentalist belief and correct the destabilising trend-following behaviour of chartists. Only by using tight leverage constraints for prices above the fundamental price, the destabilising effect of chartists can be limited. The following proposition shows that these effects hold in general for $g > R$ in the limit case $\beta = \infty$.

Proposition 3. *Consider the ABS with fundamentalists and chartists under possible demand constraints. Assume* $z_s > 0$*,* $C > 0$ *and* $\beta = \infty$*. Then if* $g > R$*, the fundamental steady state* $E_1 = (x^*, n_1^*) = (0,0)$ *is locally unstable for all* $T \geq 0$ *and* $I^{max} > p_f z_s$*. Furthermore, if price dynamics for all initial prices* $x_0 = \varepsilon$ *are bounded by some smallest interval* $[x^{min}, x^{max}]$ *with* $x^{min} < 0 < x^{max}$*, the following effects hold:*

- *(i) an increase in* T *(i.e. a tighter short-selling constraint) has no effect on* x^{min}*;*
- *(ii) an increase in* T *(i.e. a tighter short-selling constraint) has a weakly increasing effect on* x^{max}*;*
- *(iii) a decrease in* I^{max} *(i.e. a tighter leverage constraint) has a weakly decreasing effect on* x^{min}*;*
- *(iv) a decrease in* I^{max} *(i.e. a tighter leverage constraint) has a weakly decreasing effect on* x^{max}*. There is a price ceiling*

$$x^{L+} \equiv I^{max}/z_s - p_f > 0 \tag{29}$$

on the dynamics implying $x^{max} \leq x^{L+}$*. A locally stable, non-fundamental steady state* $E_2 = (x^{L+}, 0)$ *exists for sufficiently low* I^{max}*.*

Proof. See Appendix A □

The results suggest that combining the two policies cannot help to stabilise the dynamics, as long as the constraints affect the stabilising forces in the market. Above the fundamental value leverage constraints reduce mispricing and price volatility, but additional short-selling constraints can undo this effect. Below the fundamental value leverage constraints have an adverse effect, and additional short-selling taxes are irrelevant. The example of fundamentalists versus chartists thus points to a serious limitation of demand constraints in stabilising financial markets.

3.2. Fundamentalists versus optimists and pessimists

We saw that under the adaptive belief system (ABS) with costly fundamentalist versus chartist strategies, demand constraints may increase mispricing and price volatility. The question raises how robust these results are with respect to the set of boundedly rational strategies agents choose from. In this section I provide another simple example of an adaptive belief system with three agent types (see also Brock and Hommes, 1998; Hommes, 2013). The forecasting rules are:

$$E_{1,t}[x_{t+1}] = 0, \tag{30}$$

$$E_{2,t}[x_{t+1}] = +b, \tag{31}$$

$$E_{3,t}[x_{t+1}] = -b, \tag{32}$$

where $b > 0$ is a constant. The first rule is again the (stabilising) fundamentalist belief, this time without information-gathering costs. The two alternative strategies in this ABS are optimists with fixed biases above, and pessimists with a fixed belief below the fundamental value. These strategies do not require information costs either.

First I introduce a positive outside supple $z_s \geq 0$ into this adaptive belief system as in the two-type ABS. This extension is motivated by the aim of the paper to investigate demand constraints. A strictly positive supply is required to ensure that a market clearing price in general always exists (see Proposition 1). In economic terms, an important motivation to allow for strictly positive supply is that traders who sell the asset, not necessarily sell the asset short (Anufriev and Tuinstra, 2013, p. 1526).

The following proposition specifies the nature of price dynamics in the limit case $\beta = \infty$. It generalises Lemma 9 of Brock and Hommes (1998) to the case of positive supply $z_s > 0$.

Proposition 4. *Consider the ABS with fundamentalists, optimists and pessimists without demand constraints. Assume $z_s > 0$, $b > 0$, and $\beta = \infty$. Then:*

1. *for $0 < b < \frac{r_f}{R} a\sigma^2 z_s$, the non-fundamental steady state $E = (x^*, n_1^*, n_2^*, n_3^*) = (b/R, 0, 1, 0)$ exists and is the unique, globally stable steady state.*

2. *for* $b > \frac{r_f}{R} a\sigma^2 z_s$*, the system has a globally stable 4-cycle with prices* $\frac{1}{R}\{b, b, -b, -b\}$ *and price volatility* $\sigma = b/R$*.*

Proof. See Appendix A □

A difference with a market inhabited by chartists is that the range of feasible prices is limited, as the beliefs of all three strategies are fixed and bounded. More precisely, the price x_t can only fluctuate in the interval $[\frac{-b}{R}, \frac{b}{R}]$. Proposition 4 shows that fundamentalist traders cannot drive out simple rule-of-thumb strategies even though biased expectations b and $-b$ lie outside this interval, when agents have a high intensity of switching to better performing strategies. For small biases, the market converges to the optimistic steady state. For sufficiently large biases, cycles occur and the market is alternately dominated by optimists and pessimists.

For the baseline parametrisation without constraints, I follow Hommes (2013, p. 175), with a small value of supply $z_s = 0.01$. Under this parameters the bias b is sufficiently large to investigate the effect of demand constraints on cycles around the fundamental value.

Definition 2 (Baseline parametrisation under fundamentalists and biases)**.** *Parameters:* $z_s = 0.01$*,* $r_f = 0.1$*,* $a\sigma^2 = 1$*,* $\bar{y} = 1.01$ *(*$\rightarrow p_f = 10$*),* $b = 0.2$ *and* $\beta = 100$*.*

Given the finite value of β, we observe quasi-periodic cycles of prices around the fundamental value (Figure 6). The middle and lower time series in Figure 6b show the fractions and positions of all three agent types. Fundamentalists have a cyclical demand following the price fluctuations, while the demand of optimists and pessimists equals fundamentalists' demand shifted vertically by $\frac{1}{a\sigma^2}b$ and $-\frac{1}{a\sigma^2}b$ (see the optimal unconstrained investmen level (5)). As in the tops and troughs of the price cycle optimists and pessimists receive larger profits, the market is alternately dominated by either optimists of pessimists, reinforcing the cycle.

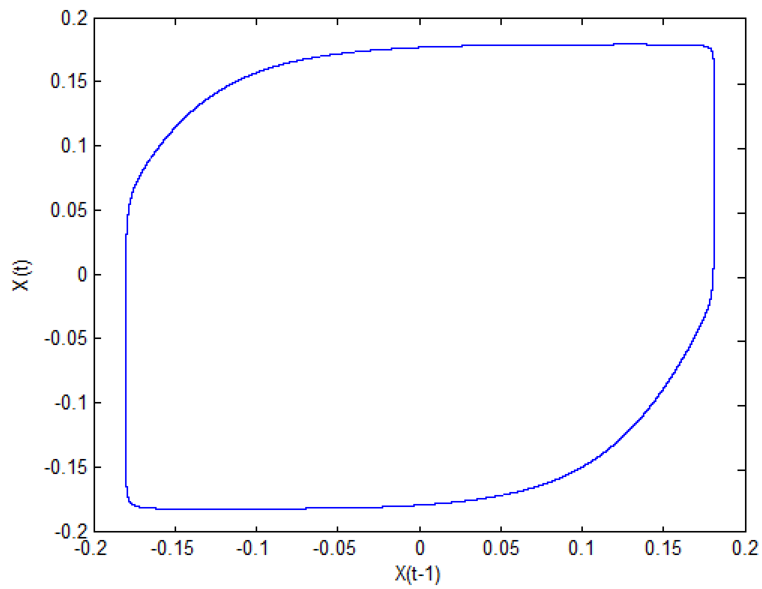
3.2.1. Short-selling constraints in the three-type market

I introduce short-selling constraints in the same way as I did in the two-type ABS. Figure 7 presents the bifurcation diagram for the short-selling tax T given the baseline parametrisation. In this three-type ABS, short-selling constraints lead a lower price volatility, and to a stable equilibrium price if T is larger than 0.06.

The stabilising effect of strict short-selling restrictions under fixed expectation rules can be understood intuitively as follows. Consider first a relatively small short-selling tax of

Figure 6: Phase plot (a) and time series (b) for the benchmark ABS with fundamentalists and opposite biases without constraints. The baseline parametrisation and $T = 0$, $I^{max} = \infty$.

(a) Phase plot showing a quasi-periodic attractor



(b) Time series. *Upper panel:* prices. *Middle panel:* fraction of fundamentalists (blue), optimists (green) and pessimists (red). *Lower panel:* positions of fundamentalists (blue), optimists (green) and pessimists (red).

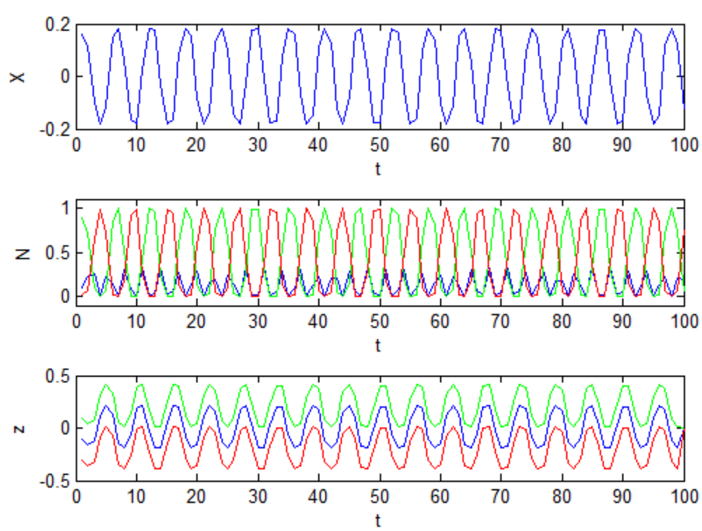
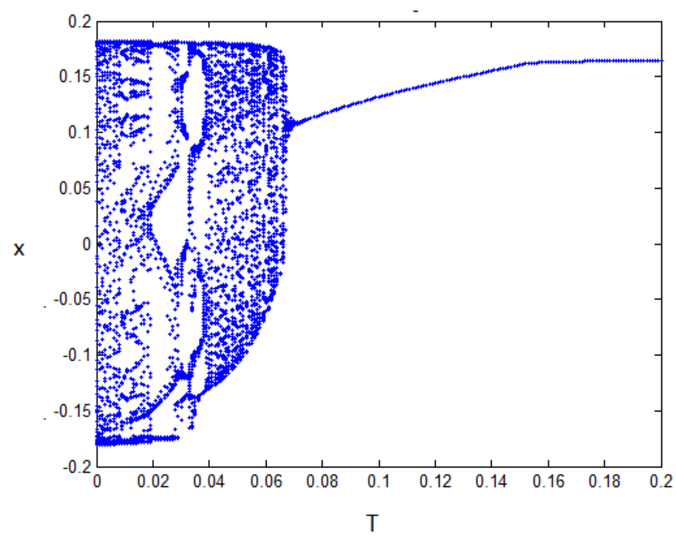


Figure 7: Bifurcation diagram for short-selling tax T given the baseline parametrisation and $I^{max} = \infty$



$T = 0.055$ as in Figure 8. The positions of agents (depicted in the lower panel Figure 8b) remain almost unchanged, but the trading costs $R\tau(z_{h,t})$ for short-selling reduce the fractions of pessimists at the price troughs, which are lower than the fractions of optimists at the price tops (see the middle panel). This in turn brings the troughs closer to the fundamental value. As the minimum price during the cycle is shifted upwards, the amplitude below the fundamental decreases.

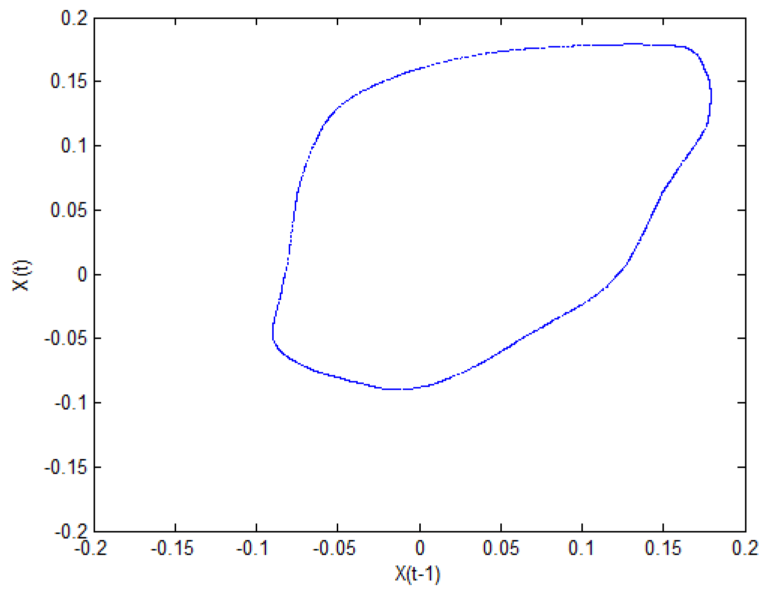
For a larger tax $T > 0.06$, the fraction of pessimists is further limited, and prices do not follow a cycle any longer, but converge to a steady state value $x = \frac{1}{R}(n_2 - n_3)b > 0$. At this price the demand of fundamentalists and optimists clears the market $n_1 z_1 + n_2 z_2 = z_s$ with $z_1 \approx 0$ because fundamentalists are also leverage constrained. The difference in performance between fundamentalists and optimists, who demand $z_2 > 0$, determine the fractions n_1 and n_2, depending on the values of z_s and β. In either case there are very few pessimists ($n_3 << n_1, n_2$). The steady state in deviation from the fundamental value is positive.

3.2.2. Leverage constraints in the three-type market

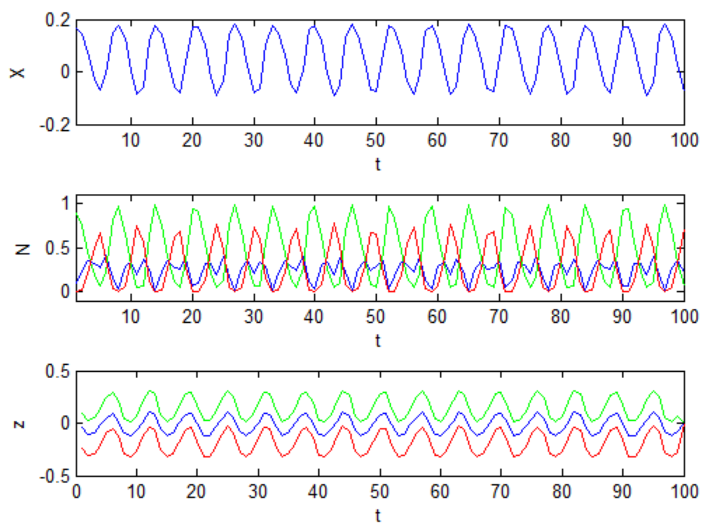
Next, I investigate the effect of leverage constraints in the three-type ABS. The parameter choices in Definition 2 require $I^{max} \geq 0.1$ to have an unconstrained fundamental value as in Assumption 1. The bifurcation diagram of the price dynamics (see Figure 9) show a opposite effect to short-selling constraints. For a leverage constraint I^{max} lower than 2.5, the maximum price during the cycle is shifted downwards and the amplitude above

Figure 8: Phase plot (a) and time series (b) for the ABS with fundamentalists and opposite biases without constraints. The baseline parametrisation and $T = 0.055$, $I^{max} = \infty$.

(a) Phase plot showing a quasi-periodic attractor

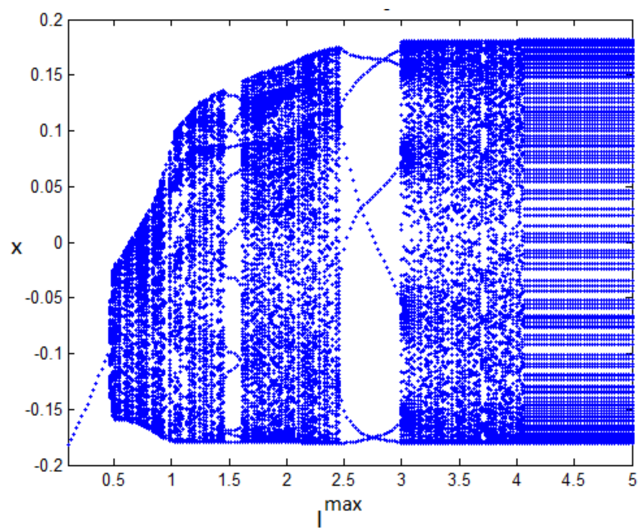


(b) Time series. *Upper panel:* prices. *Middle panel:* fraction of fundamentalists (blue), optimists (green) and pessimists (red). *Lower panel:* positions of fundamentalists (blue), optimists (green) and pessimists (red).



the fundamental decreases. Leverage constraints lead to a stable steady state below the fundamental value if they are sufficiently strict: in this case if I^{max} is smaller than 0.5.

Figure 9: Bifurcation diagram for leverage constraint I^{max} given the baseline parametrisation and $T = 0$.



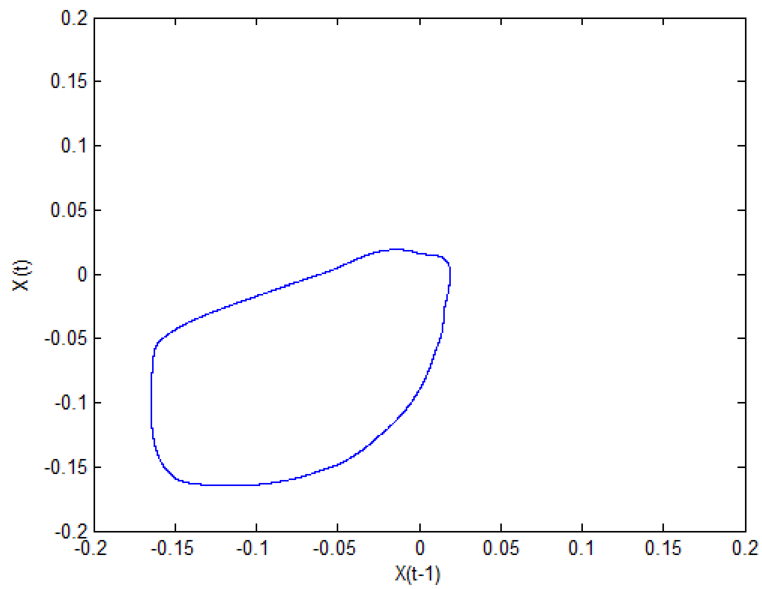
The intuition why leverage constraints stabilise the dynamics is closely related to the discussion of short-selling policies. For a sufficiently strict constraint on buying assets, the optimistic and fundamental type have a demand equal to the maximal investment level, which they maintain under small deviations in the price. These two types therefore keep approximately the same constant performance level over time and receive equal fractions. With only the pessimistic type being able to adjust his investment, prices converge to $x = \frac{1}{R}(n_2 - n_3)b < 0$ dominated by the fraction $n_3 > n_1, n_2$ of agents with a pessimists belief $-b$.

Figure 10 shows the attractor and dynamics for a maximal investment level of $I^{max} = 0.75$. For this level, optimists and fundamentalists often buy the same maximum amount, but for some periods the less optimistic view of fundamentalists leads a lower demand $z_{1,t} < z_{2,t}$. This slight difference between these two types is sufficient to stop pessimists from stabilising the market at a pessimistic steady state below the fundamental.

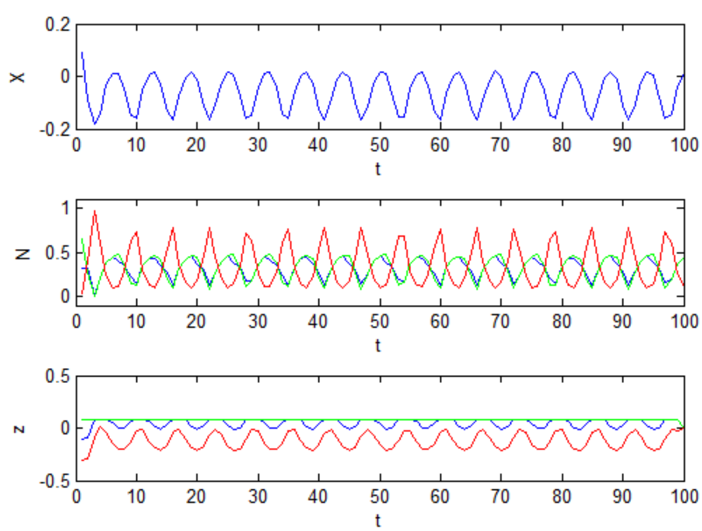
We have seen that when agents can choose from three fixed belief types, demand constraints reduce the amplitude of cycles and thus the price volatility. A sufficiently tight constraint assures that either optimistic or pessimistic beliefs have limited impact on the market price. For short-selling constraints, the impact of the pessimist belief is limited by the trading costs making it less attractive to traders. Under leverage constraints, optimists

Figure 10: Phase plot (a) and time series (b) for the ABS with fundamentalists and opposite biases without constraints. The baseline parametrisation and $T = 0$, $I^{max} = 0.75$.

(a) Phase plot showing a quasi-periodic attractor



(b) Time series. *Upper panel:* prices. *Middle panel:* fraction of fundamentalists (blue), optimists (green) and pessimists (red). *Lower panel:* positions of fundamentalists (blue), optimists (green) and pessimists (red).



(and fundamentalists as well) can take limited positions. One important difference with the market inhabited by fundamentalists and chartists is that the remaining agent type(s) cannot generate an almost self-fulfilling bubble when beliefs are fixed.

3.2.3. *Combining the two demand constraints in the three-type market*

Although demand constraints can drive down price volatility in the three-type ABS, mispricing remains when a single constraint is imposed. The reason is that although the fundamentalist belief will attract more followers in general, one type of biased traders will survive when only one constraint is imposed. For strict short-selling constraints prices converge to a level above the fundamental value, and for strict leverage constraints under the fundamental value. A natural next step is therefore to investigate the general model with short-selling and leverage constraints.

In Figure 11, the basin of attraction to a stable price is presented given the baseline parametrisation and different values of T and I^{max}. Colored regions indicate parameter values for which the dynamics converge. As expected, for loose constraints in the left upper corner of the plot the dynamics do not converge, but lead to quasi-periodic cycles as in Figures 6, 8 and 10 or periodic cycles. Imposing a single constraint by increasing T or lowering I^{max} leads to convergence to a non-fundamental steady state: a value above the fundamental for the short-selling constraint and a value below the fundamental for the leverage constraint.

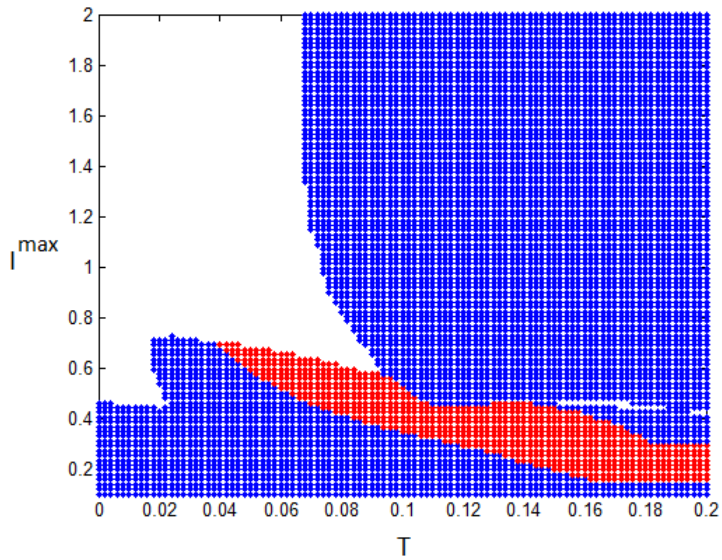
To stabilise the dynamics close to the fundamental value, the tightness of these constraints should be determined jointly. The red regions in Figure 11 show combinations of T and I^{max} that stabilise prices to a value x^* with $|x| < 0.01$ close to the fundamental. For a very high short-selling tax, leverage constraints are the single determining factor: setting I^{max} around 0.2 removes large mispricing for every value $T > 0.18$.

The possibility to stabilise the market close the fundamental price depends, however, on relative low value chosen for the intensity of choice β. The following proposition makes clear that no steady state with $x^* = 0$ exists for $\beta = \infty$.

Proposition 5. *Consider the ABS with fundamentalists, optimists and pessimists under possible demand constraints. Assume* $z_s > 0$, $b > 0$ *and* $\beta = \infty$. *Then price dynamics are bounded by* $[-b/R, b/R]$ *and price volatility is bounded by* $\sigma \leq b/R$. *If a steady state* $E = (x^*, n_1^*, n_2^*, n_3^*)$ *exists, it must have a non-fundamental price* $x^* > 0$ *for all* $T \geq 0$ *and* $I^{max} > p_f z_s$.

Proof. See Appendix A $\square$

Figure 11: Basin of attraction for different values of T and I^{max} given the baseline parametrisation. Colored regions are parameter values for which the dynamics has a stable steady state; in red regions the price converges to a value x^* with $|x^*| < 0.01$ close to the fundamental solution.



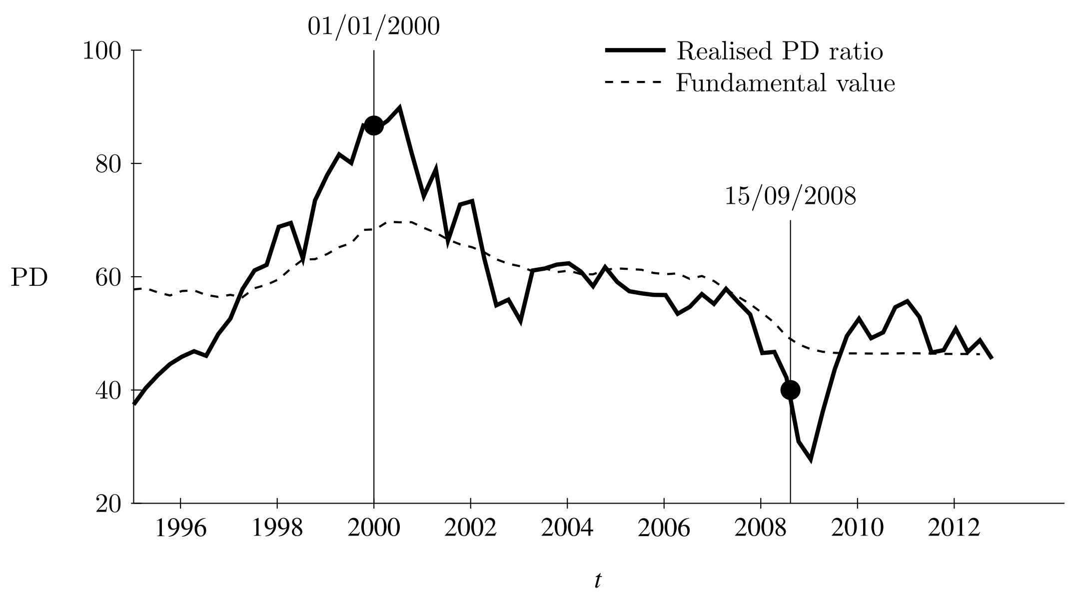
Proposition 5 shows that demand constraints weakly decrease price volatility below the level $\sigma = b/R$ that arises in the unconstrained 4-cycle for $b > \frac{r_f}{R} a\sigma^2 z_s$. On the other hand, mispricing cannot be removed by demand constraints if investors are highly sensitive to realised profits and switch rapidly between beliefs. If constraints lead to a steady state, the price must be sabove the fundamental value.

4. Discussion

The above results point to limitations of demand constraints in stabilising stylised asset markets with heterogeneous agents. In this section I discuss the policy implications of these results for the U.S. stock market. As the model is formulated in deviations from a fundamental value, I assume that policy makers dispose of a fairly accurate estimate of a fundamental stock price. In reality, there is a lot of debate about the underlying fundamental value of stock prices, and it is not self-evident that policy makers have a fully satisfactory understanding about underlying fundamentals. Nevertheless, policy makers could have concerns about large mispricing and market volatility even if the fundamental is not fully known to them, and could consider imposing demand constraints.[10]

Figure 12 plots the S&P500 price-dividend (PD) ratios from 1995Q1 to 2012Q4. As an example, the fundamental value based on the consumption-habit model of Campbell and Cochrane (1999) is plotted as well. This model has been named one of the most successful rational-agent approaches in explaining asset price behaviour by the Nobel Prize Committee (2013) in their motivation of the 2013 Nobel Prize Laureates.[11]

Figure 12: Realised PD ratio of the S&P500 and the fundamental value based on the Campbell-Cochrane consumption-habit model, 1995Q1-2012Q2.



Although many other possible fundamental values can be chosen, a robust finding is *excess volatility*, as found by Shiller (1981) and subsequently confirmed by many others. Excess volatility means that prices fluctuate much more than can be justified by their fundamentals. For the plotted PD ratios and the Campbell-Cochrane fundamental value, the most notable deviations are around 2000, the top of the dot-com bubble, and after the bankruptcy of Lehman Brothers on 15 September 2008, the height of the financial crisis.

[10] Naturally one has to take caution in applying the theoretical results in this paper directly to economic reality. Besides the fundamental value, another crucial question is what kind of rules actual investors use to forecast prices. See Hommes (2011) for a survey on asset market experiments with human subjects, which provides evidence for the use of simple rules in making investment decisions similar as in this paper.

[11] See Hommes and in 't Veld (2014) for the details of the calculation the Campbell-Cochrane fundamental value and other choices for the fundamental value.

The PD ratios of the S&P500 show the typical behaviour generated by the stylised heterogeneous agents models in this paper: recurrent bubbles and crashes and oscillating fluctuations. Hommes and in 't Veld (2014) estimate a heterogeneous agents model – closely related to model with fundamentalists and chartists in this paper – on the S&P500 PD ratios in deviations from the Campbell-Cochrane fundamental. They find that the switching of the agents between the rules has significant effects on asset prices and strongly amplifies booms and busts. I focus on the effects of demand constraints in the U.S. stock market, taking the heterogeneity of agents as given.

During the financial crisis in 2008 there have been recurrent calls for bans on short-selling, and on 19 September 2008 the Securities and Exchange Commission (SEC) indeed prohibited short-selling for 299 financial stocks. The motivation was to prevent investors from betting on a fall in stock prices, which would supposedly "restore equilibrium" to the market. On 2 October 2008 the short-selling bans ended (see Anufriev and Tuinstra, 2013, for an account of the major events from July to September 2008).

The present analysis of short-selling constraints suggests that the SEC ban is unlikely to have helped stabilising financial markets, as prices were below the fundamental value in 2008 (see Figure 12). In the two-type market with fundamentalists and chartists, prices have an endogenous lower bound of prices below the fundamental that depends on many parameters, but not on the short-selling tax T. In the three-type market with fundamentalists, optimists and chartists, price volatility can be reduced by short-selling constraints, but steady states (if they occur) are above the fundamental. So in both examples of adaptive belief systems in the asset pricing model with heterogeneous agents, short-selling constraints can *never* restore the fundamental equilibrium.

At the same time, because the price was below the fundamental in the crisis, the short-selling ban does not seem to have increased mispricing and price volatility. These adverse effects of demand constraints can happen in markets with fundamentalists and chartists, but for short-selling constraints only above the fundamental. It is therefore possible that the short-selling ban of the SEC had a negligible effect, which is supported by the observation that stock prices were falling before, during and after the short-sell ban in 2008. In general, there are considerable limitations of demand constraints in stabilising financial markets with heterogeneous agents.[12]

Despite these limitations, demand constraints can reduce mispricing and price volatility

[12]It is not clear what the effect of leverage constraints is on prices below the fundamental value. In the two-type market, price volatility increases, but in the three-type market price volatility decreases.

when prices are above the fundamental, as was the case during the dot-com bubble in 2000. In asset pricing markets with heterogeneous agents, non-fundamental steady states can occur with a price above the fundamental, where the market is dominated by investors with the most optimistic belief. If these optimistic investors are constrained borrowing money to buy stocks, prices decrease.

References

Anufriev, M., Bottazzi, G., 2006. Price and wealth dynamics in a speculative market with generic procedurally rational traders. Warwick Business School Working Paper.

Anufriev, M., Tuinstra, J., 2013. The impact of short-selling constraints on financial market stability in a heterogeneous agents model. Journal of Economic Dynamics and Control.

Brock, W. A., Hommes, C. H., 1997. A rational route to randomness. Econometrica 65 (5), 1059–1096.

Brock, W. A., Hommes, C. H., 1998. Heterogeneous beliefs and routes to chaos in a simple asset pricing model. Journal of Economic Dynamics and Control 22 (8-9), 1235–1274.

Brunnermeier, M., Oehmke, M., 2012. Bubbles, financial crises, and systemic risk. NBER Working Paper.

Buss, A., Dumas, B., Uppal, R., Vilkov, G., 2013. The equilibrium dynamics of liquidity and illiquid asset prices. Banque de France working paper.

Campbell, J. Y., Cochrane, J. H., 1999. By force of habit: A consumption-based explanation of aggregate stock market behavior. The Journal of Political Economy 107 (2), 205–251.

Chiarella, C., He, X.-Z., 2001. Asset price and wealth dynamics under heterogeneous expectations. Quantitative Finance 1 (5), 509–526.

Gaunersdorfer, A., 2000. Endogenous fluctuations in a simple asset pricing model with heterogeneous agents. Journal of Economic Dynamics and Control 24 (5), 799–831.

Geanakoplos, J., 2010. The leverage cycle. In: Acemoglu, D., Rogoff, K., Woodford, M. (Eds.), NBER Macroeconomics Annual 2009, Volume 24. University of Chicago Press, pp. 1–65.

Gromb, D., Vayanos, D., 2002. Equilibrium and welfare in markets with financially constrained arbitrageurs. Journal of Financial Economics 66 (2), 361–407.

Harrison, J. M., Kreps, D. M., 1978. Speculative investor behavior in a stock market with heterogeneous expectations. The Quarterly Journal of Economics 92 (2), 323–336.

Hommes, C., 2011. The heterogeneous expectations hypothesis: Some evidence from the lab. Journal of Economic Dynamics and Control 35 (1), 1–24.

Hommes, C., 2013. Behavioral rationality and heterogeneous expectations in complex economic systems. Cambridge University Press.

Hommes, C., in 't Veld, D., 2014. Booms, busts and behavioural heterogeneity in stock prices. CeNDEF Working Paper.

Longstaff, F. A., 2008. Flight-from-leverage in distressed financial markets. UCLA Anderson School of Management Manuscript.

Nobel Prize Committee, 2013. Understanding asset prices. Scientific background on the Sveriges Riksbank prize in economic sciences in memory of Alfred Nobel.

Shiller, R. J., 1981. Do stock prices move too much to be justified by subsequent changes in dividends? American Economic Review 71 (3), 421–36.

Simsek, A., 2013. Belief disagreements and collateral constraints. Econometrica 81 (1), 1–53.

Thurner, S., Farmer, J. D., Geanakoplos, J., 2012. Leverage causes fat tails and clustered volatility. Quantitative Finance 12 (5), 695–707.

Westerhoff, F. H., 2008. The use of agent-based financial market models to test the effectiveness of regulatory policies. Jahrbucher fur Nationalokonomie und Statistik 228 (2), 195.

Appendix A. Proofs

Proof of Proposition 1. Individual market demand given by (9) is continuous and decreasing. Aggregate demand $\sum_{h=1}^{H} n_{h,t} z_{h,t}(p_t)$ is also continuous and decreasing. It will be *strictly* decreasing as long as aggregate demand is strictly positive. Conversely, if and only if for some price p'_t every agent type h demands $z_{h,t}(p'_t) = 0$ there exists an interval of prices $I_t \subset \mathbb{R}^+$ (with $p'_t \in I_t$) such that $\sum_{h=1}^{H} n_{h,t} z_{h,t}(p_t)$ is constant (and equal to 0) for all $p_t \in I_t$. As market clearing by (10) requires $z_{h,t}(p_t) > 0$ for some $h \in H$ if $z_s > 0$, a unique equilibrium price p_t exists. □

Proof of Proposition 2. From the definition of multinomial logit fractions (15) and $\beta = \infty$, it follows that at every t the market is dominated by either fundamentalists ($n_{1,t} = 1$) or chartists ($n_{1,t} = 0$). As soon as $n_{1,t} = 1$, we get $x_{t+1} = 0$. In this fundamental state, both agent types believe $E_{h,t+j}[x_{t+j+1}] = 0$ and demand $z_{h,t+j} = z_s$, but fundamentalist pay additional costs $C > 0$. As this implies $U_{1,t} < U_{2,t}$ and thus $n_{1,t+1} = 0$, it follows that $E_1 = (0,0)$ is a steady state.

Consider an initial state $n_{1,t} = 0$ for $t = 0, 1$ and $x_{t=0} = \varepsilon$ with $\varepsilon \neq 0$ small. The price in period $t = 1$ is determined by clearing of the market dominated by chartists, i.e. the price that makes sure that chartists buy all outside supply:

$$z_{2,t}(x_t) = z_s. \tag{A.1}$$

We get $x_1 = \varepsilon(g/R)$, and $x_t = \varepsilon(g/R)^t$ as long as $n_{1,t} = 0$. For $0 < g < R$, prices converge to $E_1 = (0,0)$, and E must be globally stable.

For $g > R$, the fundamental steady state $E_1 = (0,0)$ is locally unstable. Chartists dominate the market for subsequent periods ($n_{1,t} = 0$ for $t = 1, 2, ..$) if their performance exceeds the performance of fundamentalists:

$$\begin{aligned} n_{1,t+1} = 0 \quad &\Leftrightarrow \\ U_{1,t} \quad &< \quad U_{2,t} \\ \rho_t z_{1,t-1} - C \quad &< \quad \rho_t z_{2,t-1} \end{aligned} \tag{A.2}$$

and, using the definition of ρ_t (18) and that $z_{2,t-1} = z_s$,

$$(x_t - Rx_{t-1} + a\sigma^2 z_s)(z_{1,t-1}(x_{t-1}) - z_s) - C \quad < \quad 0 \tag{A.3}$$

To find the magnitude of prices bubbles, I solve for the price x_t at which fundamentalists and chartists have an equal performance, with $x_{t-1} = (R/g)x_t$ and $x_{t-2} = (R/g)^2 x_t$:

$$\begin{aligned}
(x_t - Rx_{t-1} + a\sigma^2 z_s)(-\frac{R}{a\sigma^2}x_{1,t-1} + z_s - z_s) - C &= 0 \\
(x_t - (R^2/g)x_t + a\sigma^2 z_s)(-\frac{R^2}{a\sigma^2 g}x_t) - C &= 0 \\
(R^2/g - 1)x_t^2 - a\sigma^2 z_s x_t - a\sigma^2(g/R^2)C &= 0. \qquad \text{(A.4)}
\end{aligned}$$

If it exists, the solution to this quadratic equation is:

$$x_t = x^{b\pm} \equiv \frac{a\sigma^2 z_s \pm \sqrt{(a\sigma^2 z_s)^2 + 4(1 - g/R^2)a\sigma^2 C}}{2(R^2/g - 1)}. \qquad \text{(A.5)}$$

For all $0 < x_t < x^{b+}$, the bubble continues for at least one more period in $t+1$. The bubble ends in period $t+1$ if $x_{t+1} = (g/R)x_t > x^{b+}$. The upper bound on prices is therefore:

$$x^{max} = \frac{g}{R}x^{b+} = \frac{g}{R}\frac{a\sigma^2 z_s + \sqrt{(a\sigma^2 z_s)^2 + 4(1 - g/R^2)a\sigma^2 C}}{2(R^2/g - 1)}. \qquad \text{(A.6)}$$

Similarly the lower bound on prices is:

$$x^{min} = \frac{g}{R}x^{b-} = \frac{g}{R}\frac{a\sigma^2 z_s - \sqrt{(a\sigma^2 z_s)^2 + 4(1 - g/R^2)a\sigma^2 C}}{2(R^2/g - 1)} \qquad \text{(A.7)}$$

Finally, if no solution to (A.4) exists, the performance of chartists exceeds the performance of fundamentalists for all t, and the dynamics diverge with rate g/R. Notice that the first term in (A.4), $(R^2/g - 1)x_t^2$, is negative if $g > R^2$. Therefore no positive solution to this equation exists, proving that dynamics with positive initial prices $x_0 = \varepsilon > 0$ diverge if $g > R^2$. For a negative initial state $x_0 = \varepsilon < 0$, the existence of $x^{b\pm}$ requires

$$\begin{aligned}
(a\sigma^2 z_s)^2 + 4(1 - g/R^2)a\sigma^2 C &> 0 \\
R^2(1 + \frac{a\sigma^2 z_s^2}{4C}) &> g, \qquad \text{(A.8)}
\end{aligned}$$

completing the proof. □

Proof of Proposition 3. Observe that demand constraints cannot have an effect on the steady state $E_1 = (0,0)$. This follows because the beliefs for both types coincide with the belief of a rational representative agent ($E_{1,t}[x_{t+1}] = E_{2,t}[x_{t+1}] = 0$ for $x_t = 0$), and the rational representative agent is unconstrained by Assumption 1. Also, given any T and

$I^{max} > p_f z_s$, both types are unconstrained for $x_t = \varepsilon$ with a sufficiently small $\varepsilon \neq 0$. The steady state $E_1 = (0,0)$ is therefore locally stable if and only if $g < R$ (as in Proposition 2) with or without demand constraints.

Consider an initial state $n_{1,t} = 0$ for $t = 0,1$ and $x_{t=0} = \varepsilon$ with $\varepsilon \neq 0$ small. We get $z_{1,1} = z_s$. Chartists dominate the market for subsequent periods ($n_{1,t} = 0$ for $t = 1,2,..$) if their performance exceeds the performance of fundamentalists:

$$\begin{aligned} n_{1,t+1} = 0 & \Leftrightarrow \\ U_{1,t} & < U_{2,t} \\ \rho_t z_{1,t-1} - R\tau(z_{1,t-1}) - C & < \rho_t z_{2,t-1} - R\tau(z_{2,t-1}) \\ (x_t - Rx_{t-1} + a\sigma^2 z_s)(z_{1,t-1}(x_{t-1}) - z_s) - R\tau(z_{1,t-1}) - C & < 0 \end{aligned} \tag{A.9}$$

using that chartists never pay the short-selling tax, as $z_{2,t-1} = z_s > 0$.

Consider first the case that chartists are not leverage constrained. Then we get $x_t = \varepsilon(g/R)^t$ as long as $n_{2,t} = 1$, irrespective of possible demand constraints operative on fundamentalists. By the assumption in the proposition that price dynamics are bounded, there must exist a x_t with $x_{t-1} = (R/g)x_t$ and $x_{t-2} = (R/g)^2 x_t$ at which fundamentalists and chartists have an equal performance:

$$\begin{aligned} 0 &= (x_t - Rx_{t-1} + a\sigma^2 z_s)(z_{1,t-1}(x_{1,t-1}) - z_s) - R\tau(z_{1,t-1}(x_{t-1})) - C \\ 0 &= (x_t - \frac{R^2}{g}x_t + a\sigma^2 z_s)(z_{1,t-1}(\frac{R}{g}x_t) - z_s) - R\tau(z_{1,t-1}(\frac{R}{g}x_t)) - C, \end{aligned} \tag{A.10}$$

which implicitly defines two solutions $x^{b\pm}$. For all $x^{b-} < x_t < x^{b+}$, we have $n_{1,t+1} = 0$ and the bubble continues for at least one more period in $t+1$. The bubble ends in period $t+1$ if $x_{t+1} = (g/R)x_t \notin [x^{b-}, x^{b+}]$. The upper and lower bounds on prices are therefore $x^{max} = (g/R)x^{b+}$ and $x^{min} = (g/R)x^{b-}$.

Three effects of demand constraints follow from condition (A.10). First, at price $x_t < 0$ below the fundamental value, fundamentalists buy $z_{1,t}(x_t) > 0$, so fundamentalists pay no short-selling tax. An increase in T has thus no effect on x^{min}.

Second, the short-selling tax T has an effect on x^{max} through two channels. Notice that for $x_t > 0$ fundamentalists have a strictly lower demand $z_{1,t} < z_{2,t}$, although $z_{1,t}$ can be both positive and negative depending on the parameters. The first channel runs through the short-selling tax, which weakly increases the costs for fundamentalists to short-sell in the rising market, and reduces the performance of fundamentalists. The second channel runs through demand. A tighter short-selling constraint weakly increases fundamentalists' demand $z_{1,t}$ reduces the difference $z_{1,t-1}(x_{1,t-1}) - z_s$ in (A.10). Fundamentalists require a larger positive excess return ρ_t and thus a more positive x_t to overcome costs. An increase in T strictly decreases x^{max} if $z_{1,t}(x^{max}) < 0$, and in general weakly decreases x^{max}.

Third, a decrease in I^{max} can have a decreasing effect on x^{min} for the following reason. Notice that for $x_t < 0$ fundamentalists have a strictly higher demand $z_{1,t} > z_{2,t}$. A tighter leverage constraint weakly decreases fundamentalists' demand $z_{1,t}$, and as $z_{1,t} > z_s$ it reduces the difference $z_{1,t-1}(x_{1,t-1}) - z_s$ in (A.10). Fundamentalists require a larger positive excess return ρ_t and thus a more negative x_t to overcome the information costs C. The lower bound on prices x_{min} is weakly decreasing if I^{max} is decreased.

Finally, consider the case that chartists are leverage constrained in buying $z_s > 0$. The leverage constraint implies:

$$z_s = z_{2,t} \leq \frac{I^{max}}{p_f + x_t} \tag{A.11}$$

$$\Rightarrow x_t \leq x^{L+} \equiv \frac{I^{max}}{z_s} - p_f, \tag{A.12}$$

which imposes a *price ceiling* $x^{L+} > 0$ on the price dynamics. Consider that at $t-1$ chartists are leverage constrained leading to $x_{t-1} = x^{L+}$. Then the condition (A.9) determines whether in period t chartists dominate the market, i.e. $n_{1,t} = 0$. If $n_{1,t} = 0$ then $x_t = x^{L+}$, and it is possible to end up in the steady state $E_2 = (x^{L+}, 0)$. For $E_2 = (x^{L+}, 0)$ to be a steady state, the performance of chartists should be higher for $x_t = x_{t-1} = x^{L+}$:

$$(x^{L+} - Rx^{L+} + a\sigma^2 z_s)(z_{1,t-1}(x^{L+}) - z_s) - R\tau(z_{1,t-1}(x^{L+})) - C \quad < \quad 0. \tag{A.13}$$

Condition (A.13) holds for $x^{L+} = 0$ (but not for $x^{L+} = \infty$) so it must be fulfilled for sufficiently low x^{L+}. As $x^{L+} = I^{max}/z_s - p_f$, this implies that $E_2 = (x^{L+}, 0)$ is a steady state for I^{max} close enough to $p_f z_s$, i.e. for a sufficiently low I^{max}. □

Proof of Proposition 4. In the absence of demand constraints, the demand for each type is given by

$$z_{1,t} = \frac{1}{a\sigma^2}(-Rx_t) + z_s, \tag{A.14}$$

$$z_{2,t} = \frac{1}{a\sigma^2}(b - Rx_t) + z_s, \tag{A.15}$$

$$z_{3,t} = \frac{1}{a\sigma^2}(-b - Rx_t) + z_s. \tag{A.16}$$

In every period t we have $z_2 > z_1 > z_3$. Realised profits are $U_{h,t} = \rho_t z_{h,t}$. This implies either

$U_{2,t} > U_{1,t} > U_{3,t}$ or $U_{3,t} > U_{1,t} > U_{2,t}$ for all $\rho_t \neq 0$. If $\rho_t = 0$ for $t = 0$, we have $U_{h,0} = 0$ for all h leading to $n_{h,1} = 1/3$ and $x_1 = 0$. However, the fundamental state cannot be a steady state, as $\rho_t > 0$ for $x_t = x_{t-1} = 0$. So for $\beta = \infty$ the market is always dominated at $t = 1$ by either optimists or pessimists.

Consider an initial state $n_{3,t=0} = 1$ and $x_{t=0} = -b/R$ with $\rho_{t=0} < 0$. Then in period $t = 1$ pessimists dominate the market, implying $n_{3,1} = 1$ and $x_1 = -b/R$. This pessimistic state can not be a steady state, as excess returns at $t = 1$

$$\rho_1 = -b/R - b + a\sigma^2 z_s > 0, \tag{A.17}$$

are strictly positive, implying $U_{2,1} > U_{1,1} > U_{3,1}$. This leads to an optimistic state $n_{2,2} = 1$ and $x_2 = b/R$. Excess returns at $t = 2$ are

$$\rho_2 = b/R + b + a\sigma^2 z_s > 0, \tag{A.18}$$

and strictly positive, maintaining the optimistic state $n_{2,3} = 1$ and $x_3 = b/R$. At $t = 3$ excess returns are

$$\rho_3 = b/R - b + a\sigma^2 z_s. \tag{A.19}$$

As this expression can be either positive or negative, two possibilities arise for the dynamics depending on b. For low biases $0 < b < \frac{r_f}{R} a\sigma^2 z_s$, excess returns remain positive in the optimistic state, and $E = (b/R, 0, 1, 0)$ is the globally stable steady state.

For large biases supply $b > \frac{r_f}{R} a\sigma^2 z_s$, excess returns ρ_3 are strictly negative, returning the price to the negative state $n_{3,4} = 1$ and $x_4 = -b/R$. Under this low level of outside supply

$$\rho_4 = -b/R - b + a\sigma^2 z_s < 0, \tag{A.20}$$

and $t = 4$ we are in back in the initial pessimistic state started in $t = 0$. The result is a stable 4-cycle with prices $\frac{1}{R}\{b, b, -b, -b\}$. The average price is $\bar{x} = 0$ and the price volatility (the square root of the variance) along the cycle is b/R. □

Proof of Proposition 5. Under general demand constraints, we have $z_2 \geq z_1 \geq z_3$, where the inequality is strict unless some types are both leverage constrained at $z_{h,t} = I^{max}/(p_f + x_t)$, or short-selling constrained at $z_{h,t} = 0$. As aggregate demand is decreasing in the price, market clearing implies that prices are maximal when optimists dominate the market ($z_{2,t}(x_t) = z_s$), and minimal when pessimists dominate the market ($z_{3,t}(x_t) = z_s$). As demand constraints pushes large investment positions in the direction of $z_{h,t} = 0$, price dynamics must be bounded by the interval $[-b/R, b/R]$.

From these price bounds it is straightforward to derive the maximal price volatility σ. By definition the price variance is given by

$$\sigma^2 = \int_{-b/R}^{b/R} (x_t - \bar{x})^2 f(x_t) dx_t, \tag{A.21}$$

where $\bar{x}$ is the average price and $f(x_t)$ is the probability density function for the price x_t. Because of convexity of the function $(x_t - \bar{x})^2$, the price variance is maximised when there are only two price states at the bounds:

$$f(x_t) = \begin{cases} \frac{1}{2} & \text{if } x_t = -b/R \\ 0 & \text{if } x_t \in (-b/R, b/R) \\ \frac{1}{2} & \text{if } x_t = b/R, \end{cases} \tag{A.22}$$

which gives $\sigma^2 = (b/R)^2$ and price volatility $\sigma = b/R$. For any other distribution of prices $f(x_t)$, price volatility is strictly lower $\sigma < b/R$.

Now consider a possible steady state $E = (x^*, n_1^*, n_2^*, n_3^*)$. It will be shown that $x^* > 0$ must hold using a proof by contradiction. Assume there is a steady state with $x^* \leq 0$. In this steady state, excess returns are strictly positive: $\rho^* = x^* - Rx^* + a\sigma^2 = -r_f x^* + a\sigma^2 > 0$. Because $z_2 \geq z_1 \geq z_3$, optimists have the highest performance $U_2 \geq U_1 \geq U_3$ and the steady state has either $n_2^* = 1$, or $n_2^* = n_h^*$ for $h = 1$ and possibly $h = 3$. If optimists dominate ($n_2^* = 1$), the price x^* is such that $z_2(x^*) = z_s$. However, $z_2(0) > z_s$ and demand is decreasing in x_t, contradicting the assumption $x^* \leq 0$. If optimists coexist with other types ($n_2^* = n_h^*$ for some $h \neq 2$), it must be the case that the types have equal performance $U_2 = U_h$. But then they must be leverage constrained at $z_2(x^*) = z_h(x^*) = I^{max}/(p_f + x^*)$. Market clearing implies $x^* = I^{max}/z_s - p_f > 0$, also contradicting the assumption of a steady state price equal to or below the fundamental price.

□