Increasing Borrowing Costs and the Equity Premium

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Abstract

Simulating a realistic-sized equity premium in macroeconomic models has proved a daunting challenge, hence the "equity premium puzzle". "Resolving" the puzzle requires heavy lifting. Precise choices of particular preferences, shocks, technologies, and hard borrowing constraints can do the trick, but haven't stopped the search for a simpler and more robust solution.

This paper suggests that soft borrowing costs, which are rapidly rising in the amount borrowed, imbedded in an otherwise standard OLG model can work. Its model features isoelastic preferences with modest risk aversion, Cobb-Douglas production, realistic shocks, and reasonable fiscal policy. Absent borrowing costs, the model's equity premium is extremely small. Adding the costs readily produces large equity premiums.

These results echo, but also differ from those of Constantinides, Donaldson, and Mehra (2002). In their model, hard borrowing constraints on the young can produce large equity premiums. Here soft, but rising borrowing costs on all generations are needed.

The solution method builds on Hasanhodzic and Kotlikoff (2013), which uses Marcet (1988) and Judd, Maliar, and Maliar (2009, 2011) to overcome the curse of dimensionality.

Keywords: Equity Premium; Borrowing Constraints; Aggregate Shocks; Incomplete Markets; Stochastic Simulation.

JEL Classification: E62, H55, H31, D91, D58, C63, C68

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1 Introduction

The U.S. equity premium—the difference between the mean return on U.S. stocks (the market portfolio) and short-term Treasuries (3-month T-Bills)—has averaged 6 percent on an annual basis, since 1925. The average risk-free rate during this period has averaged 1 percent. In their seminal paper Mehra and Prescott (1985) showed that a reasonably calibrated representative-agent model produces an equity premium of at most 0.35 percent and a risk-free rate of 4 percent—hence the "equity premium puzzle".

A large literature has tried to resolve the puzzle, invoking hard borrowing constraints (Constantinides, Donaldson, and Mehra (2002)), alternative preferences (e.g., Constantinides (1990), Abel (1990), Epstein and Zin (1991), Jermann (1997), Campbell and Cochrane (1999), Campbell (2001), Ju and Miao (2012), and Liu and Miao (2014)), rare disasters in technology processes (e.g., Rietz (1988) and Barro (2006)), and behavioral models, such as those based on prospect theory (e.g., Barberis, Huang, and Santos (2001)). See also the survey by Mehra (2006) and the references therein.

This paper shows that soft borrowing costs, which are rapidly rising in the amount borrowed, imbedded in an otherwise standard general equilibrium OLG model can generate a sizable equity premium. The model features ten periods, isoelastic preferences with risk aversion of 2, Cobb-Douglas production, and realistically calibrated TFP shocks. It also includes government consumption at 20 percent of GDP and pay-as-you-go social security. The model features no government debt, but can be relabeled (see Green and Kotlikoff, 2008) to have as much or as little debt as desired.

Absent borrowing costs, the model's equity premium is extremely small. The borrowing costs reduce the private supply of bonds dramatically, increasing the bond price and lowering the safe rate of return. The equity premium rises from essentially zero without borrowing costs to over 5 percent with them, and the risk-free rate declines from roughly 6 – 8 percent to roughly 1 – 2.5 percent.

Although it is not needed to generate equity premium, including modest capital depreciation shocks or capital adjustment costs allows the model to emit a pattern of increasing bond holdings by age. Without either of these features, the old are the borrowers (the suppliers), with them they are the lenders (the demanders) of bonds. This is because capital depreciation shocks and capital adjustment costs expose investment principal to risk. Since, for the elderly, the only source of income is their investments, they demand bonds when the principal is risky, otherwise they supply them.

The model builds on Hasanhodzic and Kotlikoff (2013) by adding borrowing costs, stochastic depreciation, and capital adjustment costs. Like that paper's, its solution method relies on Judd, Maliar, and Maliar's (2009, 2011) algorithm—a numerically stable version of Marcet (1988)—to overcome the curse of dimensionality.

Comparison to Constantinides, Donaldson, and Mehra (2002)

Constantinides, Donaldson, and Mehra (2002) posit a three-period, partial equilibrium OLG model with pure exchange and a hard borrowing constraint on the young to resolve the equity premium puzzle. They justify this constraint based on the inability of the young to borrow against their future earnings. The authors suggest that their results would also hold in more realistic extensions, as long as the young face hard borrowing constraints. The extensions they mention include a larger number of generations, standard technology, and the inclusion of government policy.

Recall that this paper features ten generations, Cobb-Douglas technology, and government spending and generational policy. But it assumes soft, but rapidly rising borrowing costs, which seem more realistic (see the next subsection). Interestingly, limiting the borrowing costs to a subset of generations does not have the effect of lowering the risk-free rate and increasing the equity premium. Only when *all* generations face the costs does the supply of bonds become limited enough for the risk-free rate to decline substantially and the equity premium to emerge.

Borrowing Costs

The model's borrowing costs are implemented via a smoothing function proposed by Chen and Mangasarian (1996). The function is smooth and rising for negative bond holdings, and is essentially zero when bond holdings are close to zero or positive (see Figure 1).

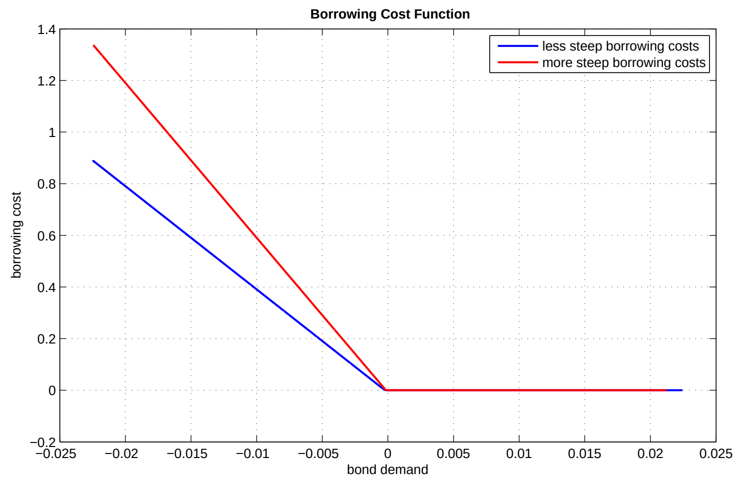

Figure 1: Borrowing cost functions with different slope parameters for a fixed level of assets. The x-axis displays bond demands. Since the asset level is fixed, the functions are increasing in bond shares.

This representation seems a reasonable alternative to hard borrowing constraints, which prevent borrowing altogether. Most households appear able to borrow very small amounts at low rates. But taking out more sizable loans on an unsecured basis entails credit card-level interest rates.

As one would expect, the steeper is the borrowing cost function, the smaller is the gross bond supply, the lower is the risk-free rate, and the larger is the equity premium. For example, in the equilibrium with the risk-free rate of roughly 1 – 2.5 percent and the equity premium of roughly 5 percent, the ratio of the marginal borrowing cost to the risk-free rate is typically 10 to 1. But in the equilibrium where this ratio is typically 2 to 1, the risk-free rate and the equity premium are roughly 3 – 4.5 percent and 3 percent, respectively.

Government Debt

Recall that the model features no government debt. This is done to conveniently isolate private bond transactions, which, unlike the government, are subject to borrowing costs. But the model can be relabeled to produce any time path of exogenously specified debt values without affecting the equilibrium (see Kotlikoff (1986, 1988, 1993, 2003), Auerbach and Kotlikoff (1987), and Green and Kotlikoff (2008)).[1]

Since with a different set of words the model can be described as featuring any amount of government debt (and associated taxes and transfers), the level of government debt per se does not affect the equity premium. What impacts the equity premium is the structure of the model, including borrowing costs as well as government spending and generational policies.

The paper is organized as follows. Sections 2 and 3 describe the model and its calibration. Section 4 presents results, Section 5 conducts sensitivity analysis, and Section 6 evaluates the accuracy of solutions. Section 7 concludes. The solution algorithm is relegated to the Appendix.

[1] To see this, recall that the model features the intergenerational redistribution policy which takes resources from the workers and gives them to the elderly. Now, whatever amount, X_t, is taken from the workers in period t can be relabeled as a borrowing of X_t (or any fraction of it, including a fraction above 1) at time t by the government with repayment at time $t+1$ of $X_t \times (1+\bar{r}_t)$ plus a tax on the workers of $X_t \times (1+\bar{r}_t)$ at time $t+1$, where $\bar{r}_t$ is the return on the bond purchased at time t.

If X_t (the amount said to be borrowed) exceeds the amount of taxes, Z_t, being collected at time t (with the no-debt labeling), the government would describe this as borrowing X_t at time t, making a transfer payment of $X_t - Z_t$ to the worker at time t, having the worker receive X_t plus interest at $t+1$, but having the worker pay in taxes, at $t+1$, the amount Z_t plus interest, plus $X_t - Z_t$ plus interest. Note that with this as with any other relabeling, on balance and ignoring the labels, the worker hands over Z_t at time t to the government and gets back, on balance, zero at time $t+1$.

To have the elderly also holding bonds, the government would say that at time t the elderly are buying M_t in bonds and receiving a transfer payment at time t of M_t. At time $t+1$, they are receiving $M_t \times (1+\bar{r}_t)$ in principal plus interest, but also paying a tax of this same amount. So the government takes nothing extra on net from the elderly at time t and at time $t+1$, but gets to announce extra debt outstanding at time t of M_t.

2 The Model

The model features G overlapping generations with shocks to total factor productivity and either capital depreciation shocks or capital adjustment costs. Each agent works full time through retirement age R, dies at age G, and maximizes expected lifetime utility. There is an increasing cost of supplying bonds, i.e. of borrowing. If there are adjustment costs, firms maximize their financial value, i.e. the present value of their revenue flow, otherwise they maximize static profits.

2.1 Endowments and Preferences

The economy is populated by G overlapping generations that live from age 1 to age G. All agents within a generation are identical and are referenced by their age g and time t. Each cohort of workers supplies 1 unit of labor each period. Hence, total labor supply equals the retirement age R. Utility is time-separable and isoelastic, with risk aversion coefficient γ:

$$u(c) = \frac{c^{1-\gamma} - 1}{1 - \gamma}. \tag{1}$$

2.2 Technology

Production is Cobb-Douglas with output Y_t given by

$$Y_t = z_t K_t^{\alpha} L_t^{1-\alpha}, \tag{2}$$

where z is total factor productivity, α is capital's share of output, and L_t is labor demand, which equals R, labor supply. Equilibrium factor prices are given by

$$w_t = z_t(1-\alpha)\left(\frac{\sum_{g=1}^{G}\theta_{g,t-1}}{R}\right)^{\alpha}, \tag{3}$$

$$r_t = z_t\alpha\left(\frac{\sum_{g=1}^{G}\theta_{g,t-1}}{R}\right)^{\alpha-1} - \delta_t, \tag{4}$$

where depreciation $\delta_t \sim \mathcal{N}(\mu_\delta, \sigma_\delta^2)$, as in Ambler and Paquet (1994).

With capital adjustment costs r is given by

$$r_t = \frac{z_t\alpha(\frac{K_t}{R})^{\alpha-1} + \frac{m}{2}(\frac{I_t}{K_t})^2 + q_t - q_{t-1}}{q_{t-1}}, \tag{5}$$

where $q_t = 1+m\frac{I_t}{K_t}$ is the price of capital, $K_t = \frac{\sum_{g=1}^{G}\theta_{g,t-1}}{q_{t-1}}$ is the capital stock, $I_t = K_{t+1}-K_t$ is the investment at time t, and m is the adjustment cost parameter.

Total factor productivity, z, obeys

$$\ln(z_{t+1}) = \rho\ln(z_t) + \epsilon_{t+1}, \tag{6}$$

where $\epsilon_{t+1} \sim \mathcal{N}(0, \sigma^2)$.

2.3 Financial Markets

Households save and invest in either risky capital or one-period safe bonds. Investing 1 unit of consumption in bonds at time t yields $1+\bar{r}_t$ units in period $t+1$. The safe rate of return, $\bar{r}_t$, is indexed by t since it is known at time t although it is received at time $t+1$. The total demand for assets of household age g at time t is denoted by $\theta_{g,t}$, and its share of assets invested in bonds is denoted by $\alpha_{g,t}$. Households enter period t with $\theta_{g-1,t-1}$ in assets, which corresponds to the total assets they demanded the prior period. Since investment decisions are made at the end of the period, the aggregate supply of capital in period t, K_t, is the sum of assets brought by the households into period t, i.e.

$$K_t = \sum_{g=1}^{G} \theta_{g,t-1}, \tag{7}$$

normalized by q_{t-1} in the case of adjustment costs. Bonds are in zero net supply, hence by being short (long) bonds, households are borrowing (lending) to each other.

2.4 Borrowing Costs

Suppliers of bonds incur borrowing costs which are increasing in the amount of bonds supplied. To borrow the amount of $\alpha\theta$ households have to pay the borrowing cost of $f(\alpha)\theta$, where

$$f(\alpha) = 0.2\left(-b\alpha - 1 + \frac{1}{5}\ln(1 + \mathrm{e}^{5b\alpha+5})\right) \tag{8}$$

and b is the parameter described in Section 3 governing slope of f. Since f is increasing in bond shares (α), for a given level of assets (θ) the marginal borrowing cost is increasing in total amount borrowed ($\alpha\theta$).

2.5 Government

Government spending equals a fixed share ξ of GDP each period. It is financed by the payroll tax τ. Two types of intergenerational redistribution policy are considered: a fixed benefit policy and a proportional tax policy. Under the former, the government takes a variable share of the wage from the workers and gives a fixed benefit to the elderly. Under the latter, it takes a fixed share of the wage from the workers and gives a variable amount to the elderly. The transfers to the elderly are denoted by H_t.

2.6 Household Problem

Households of age g in state (s, z, δ) maximize expected remaining lifetime utility given by

$$V_g(s, z, \delta) = \max_{c,\theta,\alpha} \left\{u(c) + \beta \mathrm{E}\left[V_{g+1}(s', z', \delta')\right]\right\} \tag{9}$$

subject to

$$c_{1,t} = \ell_1(1 - \tau_t)w_t - \theta_{1,t} + (1 - \ell_1)H_t, \tag{10}$$

$$\begin{aligned} c_{g,t} = {} & \ell_g(1 - \tau_t)w_t + \left[\alpha_{g-1,t-1}(1 + \bar{r}_{t-1}) + (1 - \alpha_{g-1,t-1})(1 + r_t)\right]\theta_{g-1,t-1} - \theta_{g,t} \\ & - f(\alpha)\theta_{g-1,t-1} + (1 - \ell_g)H_t, \\ & \text{for } \ 1 < g < G, \ \text{ and} \end{aligned} \tag{11}$$

$$\begin{aligned} c_{G,t} = {} & \ell_G(1 - \tau_t)w_t + \left[\alpha_{G-1,t-1}(1 + \bar{r}_{t-1}) + (1 - \alpha_{G-1,t-1})(1 + r_t)\right]\theta_{G-1,t-1} \\ & - f(\alpha)\theta_{G-1,t-1} + (1 - \ell_G)H_t, \end{aligned} \tag{12}$$

where $c_{g,t}$ is the consumption of a g-year old at time t, τ_t is the payroll tax financing government spending and transfers to the elderly, H_t is the benefit given to the elderly, and (10)–(12) are budget constraints for age group 1, those between 1 and G, and that for age group G.

2.7 Equilibrium

At time t, the economy's state is (s_t, z_t, δ_t), with $s_t = (\theta_{1,t-1}, \ldots, \theta_{G-1,t-1})$ denoting the set of age-specific asset holdings. Given the initial state of the economy s_0, z_0, δ_0, where $s_0 = (\theta_{1,-1}, \ldots, \theta_{G-1,-1})$, the recursive competitive equilibrium is defined as follows:

Definition. The recursive competitive equilibrium is governed by the collection of the value functions and the household policy functions for total savings $\theta_g(s, z, \delta)$, the share of savings invested in bonds $\alpha_g(s, z, \delta)$, and consumption $c_g(s, z, \delta)$ for each age group g, the choices for the representative firm $K(s,z,\delta)$ and $L(s,z,\delta)$, as well as the pricing functions $r(s,z,\delta)$, $w(s,z,\delta)$, and $\bar{r}(s,z,\delta)$ such that:

1. Given the pricing functions, the value functions (9) solve the recursive problem of the households subject to the budget constraints (10)–(12), and θ_g, α_g, and c_g are the associated policy functions for all g and for all dates and states.

2. Wages and rates of return on capital satisfy (3) and (4) or (5), i.e. at each point, for given w and r the firm maximizes profits if there are no adjustment costs and maximizes firm value otherwise.

3. All markets clear: Labor and capital market clearing conditions are implied by $L_t = R$ and (7). Since bonds are in zero net supply, bond market clearing requires

$$\sum_{g=1}^{G} \alpha_g(s,z,\delta)\theta_g(s,z,\delta) = 0. \tag{13}$$

   Market clearing conditions in labor, capital, and bond markets and satisfaction of household budgets imply market clearing in consumption.

4. The government balances its budget, i.e.,

$$\tau_t = \frac{\xi Y_t + H_t(G-R)}{w_t R}. \tag{14}$$

Finally, for all age groups $g = 1, \dots, G-1$, optimal intertemporal consumption and investment choice satisfies

$$1 = \beta \mathrm{E}_z \big[\big(1 + r(s', z', \delta') + \alpha_g(s, z, \delta)(\bar{r}(s, z, \delta) - r(s', z', \delta')) \qquad (15)$$
$$- f(\alpha_g(s, z, \delta))\big) \frac{u'(c_{g+1}(s', z', \delta'))}{u'(c_g(s, z, \delta))} \big]$$

and

$$0 = \mathrm{E}_z \left[u'(c_{g+1}(s', z', \delta'))(\bar{r}(s, z, \delta) - r(s', z', \delta') - f'(\alpha_g(s, z, \delta))) \right], \qquad (16)$$

where E_z is the conditional expectation of z' given z, and $f'(\alpha) = 0.2b\left(-1 + \frac{\mathrm{e}^{5b\alpha+5}}{1+\mathrm{e}^{5b\alpha+5}}\right)$ is the derivative of f given by (8). Note that the endogenous part of the state next period, s', is determined by the asset demands chosen the period before. Hence, the only stochastic element of the next period's state vector is the aggregate productivity level, z'.

3 Calibration

The parameters are calibrated as follows.

3.1 Endowments and Preferences

We set γ to 2. Agents work for 7 periods and live for 10. Hence, each period represents 6 years. We set the quarterly subjective discount factor, β, at 0.99, as is standard in the macroeconomics literature.

3.2 Technology

Quarterly values for ρ and σ are 0.95 and 0.01, respectively, which is right in the middle of empirical estimates (see, e.g., Hansen (1985) or Prescott (1986)). Capital share of output, α, equals 0.33. In the model with stochastic depreciation, quarterly values for μ_δ and σ_δ are

0.0123 and 0.0026, respectively. In the model with adjustment costs, as well as in the base model without either stochastic depreciation or adjustment costs, depreciation is zero. The adjustment cost parameter m is set to 10.

3.3 Borrowing Costs

The borrowing cost parameter b equals 200 in the flatter case and 300 in the steeper case.

3.4 Government

The government spending share, ξ, equals 20 percent. Under the fixed benefit policy, the benefit equals 20 percent of the average wage. Under the proportional tax policy, the tax equals 20 percent of the wage.

4 Results

4.1 Fluctuations

Before discussing the demand for bonds and the associated equity premium, it is instructive to describe the fluctuations against which the agents might want to insure. Figure 2 plots the evolution over 640 years of the capital stock, output, the wage, and the rate of return on capital for three cases—the model with stochastic depreciation, the model with adjustment costs, and the base model without either. The fixed benefit policy is in place in each case.

Total factor productivity z across the 640 years has a mean of 0.999 (1.000) and a standard deviation of 0.033 (0.032) in models without (with) stochastic depreciation. This produces sizable fluctuations in capital, output, the wage, and the annualized rate of return, with standard deviations of 0.136, 0.203, 0.019, and 0.003 around the means of 3.108, 5.356, 0.513, and 0.078 in the base model.[2]

[2] The first 50 observations are excluded from computations of statistics in this section so that results are insensitive to initial conditions.

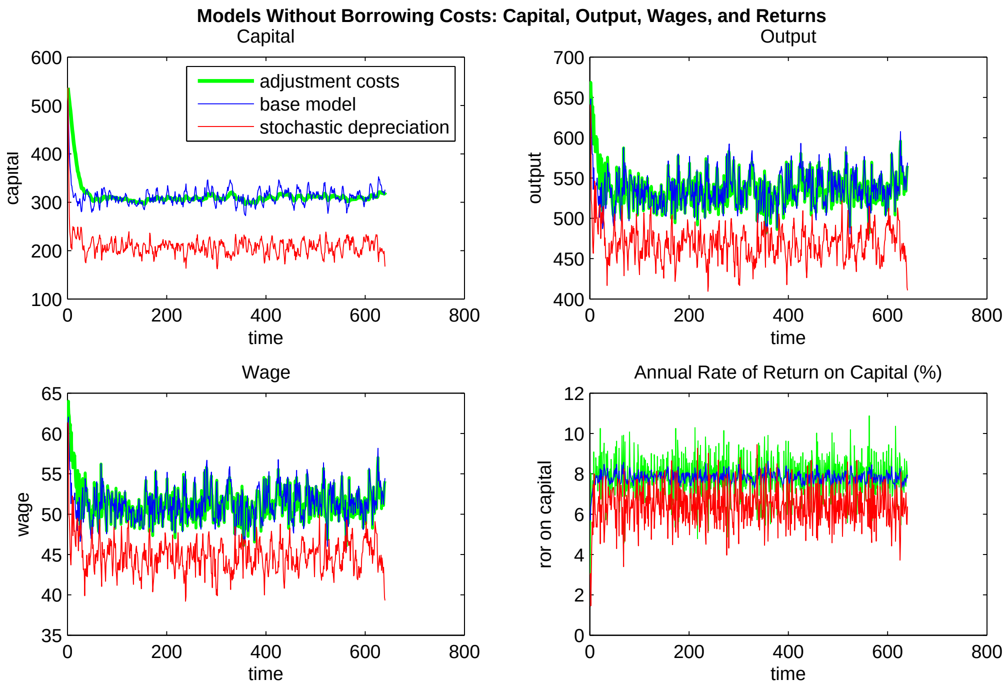

Figure 2: Capital, output, wage, and rate of return on capital in the base model, and in models with stochastic depreciation or adjustment costs, with the fixed benefit policy and without borrowing costs.

Over the same time period stochastic depreciation has a mean of 0.293 and a standard deviation of 0.064. It has the effect of lowering the capital stock to 2.063 on average and increasing its standard deviation somewhat to 0.138. Output and the wage decrease to 4.673 and 0.447 on average. The rate of return on capital is directly hit by depreciation shocks, hence its standard deviation triples to 0.009 around the mean of 0.064.

With adjustment costs firms "smooth" or "partially adjust" their investment behavior over time. Hence, the standard deviation of capital, 0.048, is less than half that in the base model around roughly the same mean (3.083). Consequently, output and the wage fluctuate less than in the base model: the standard deviation is 0.181 around the mean of 5.342 for the output, and 0.017 around the mean of 0.511 for the wage. The rate of return on capital is even more volatile than in the base model—it has a standard deviation of 0.010 around the mean of 0.078—since the marginal cost of investment, q, fluctuates substantially, exhibiting a standard deviation of 0.048 around the mean of 1.001.

4.2 Bond Demands and the Equity Premium Without Borrowing Costs

Figure 3 plots age-specific average bond shares, assets, and bond demands for the three models without borrowing costs. The bond share is the proportion of assets invested in bonds, and the bond demand is the absolute amount demanded in bonds. This figure shows that bonds are supplied by the young in the model with stochastic depreciation, by the middle aged in the model with adjustment costs, and by the old in the base model.

This pattern is intuitive since the old live off their equity income, i.e., the assets they have accumulated (the principal) and the return earned on them. With stochastic depreciation or adjustment costs, both the principal and the return on assets are uncertain and, thus, can be lost. Thus, the old demand bonds.

On the other hand, the young live off both the wage income and, except for the first

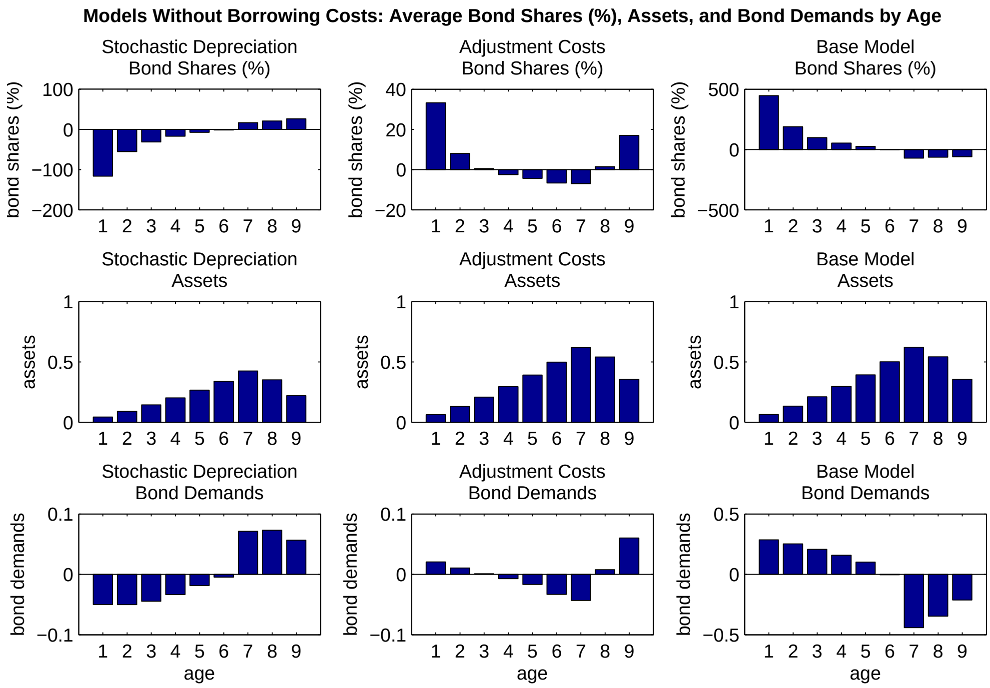

Figure 3: The average bond shares (%), assets, and bond demands by age in the models with stochastic depreciation or adjustment costs, and in the base model, with the fixed benefit policy and without borrowing costs.

generation, the equity income. With stochastic depreciation, the wages and the equity returns are uncorrelated, exhibiting a correlation coefficient of -2.77 percent. Hence, the income sources of the young are more diversified than those of the old. Consequently, the young are in a position to insure the old by selling bonds to them and going long capital (stocks). Indeed, this model exhibits a pattern of increasing bond shares with age.

With adjustment costs, the income sources of the young are less diversified: the wages and the returns exhibit a correlation coefficient of 45.07 percent. This makes the young less willing to supply bonds. In fact, the first three generations demand bonds, while the relatively wealthier middle aged supply them.

In the base model, the age pattern of bond holdings is reversed, with the old supplying the bonds that the young demand. As above, the main asset of the young—their wages—is positively correlated with the return to stocks, exhibiting a correlation coefficient of 38.97. But without stochastic depreciation or adjustment costs, the old face less risks: while the return on their assets is uncertain, the principal cannot be lost and, thus, is safe. Consequently, the old are in a position to insure the young against productivity shocks by selling bonds to them and going long capital (stocks).

The bond market provides effective insurance. For example, in the base model, it covers about one-third of the young's potential loss in wages that might arise due to an adverse shock.[3]

Figure 4 plots the demands for bonds of different age groups in specific states of the world, as characterized by good or bad z's and δ's. It shows that the bond demands are

[3] To see this, note that the young short stocks to insure against an adverse shock in z and the resulting decline in wage. Consider two scenarios. In one, the beginning-of-period capital is equal to the average capital stock over 640 periods, 5.3600, and z is equal to the average z, 0.9999, implying a wage of 0.6134 and a rate of return on capital of 0.3946 per period. In the other, capital is again equal to its average value and z is one standard deviation below average, at 0.9668, implying a wage of 0.5931 and a rate of return on capital of 0.3815. One measure of the young's potential loss in wages is the difference in wage between the two scenarios, 0.0203. Since the difference in the rates of return on capital between the two scenarios is 0.0131 and the average bond demand of the youngest age group is 0.4801, the youngest gain 0.0063 in consumption units when the adverse shock hits. Hence the potential capital gain covers about one-third of the loss.

responsive to different economic conditions. For example, the one-year-olds supply more than twice as many bonds and the seven-year-olds demand three times as many bonds in the state associated with favorable realizations of the shocks.

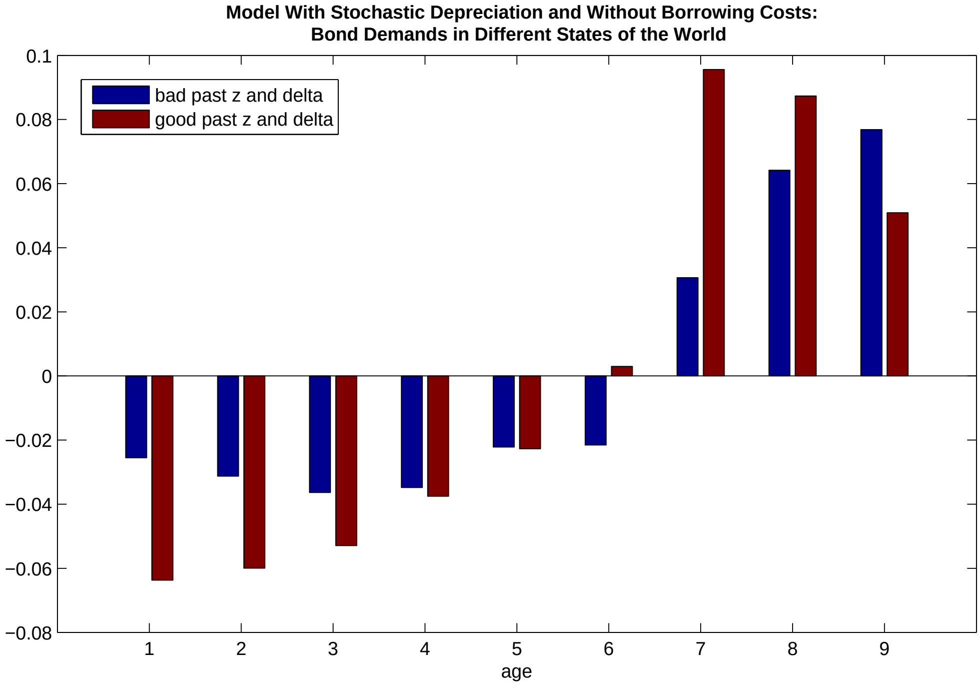

Figure 4: The average bond demands by age starting from good and bad states of the world as characterized by the z's and the δ's in the model with stochastic depreciation, the fixed benefit policy, and without borrowing costs.

The top panel of Table 1 shows that without borrowing costs, the annual equity premium is very small. For example, in the model with stochastic depreciation, it is a mere 0.021 percent. The stock returns average 6.439 percent annually, which is reasonable. However, at 6.418 percent, the bond returns are almost as high. The other two models exhibit similar patterns.

Statistic (Annual, in Percent)	Stochastic Depreciation	Adjustment Costs	Base Model
	No Borrowing Costs		
Equity Premium	0.021	0.024	0.005
Mean Stock Return	6.439	7.835	7.797
Mean Bond Return	6.418	7.811	7.793
	Flatter Borrowing Costs		
Equity Premium	3.579	3.332	3.309
Mean Stock Return	6.440	7.838	7.826
Mean Bond Return	2.860	4.506	4.517
	Steeper Borrowing Costs		
Equity Premium	5.615	5.205	5.179
Mean Stock Return	6.441	7.837	7.827
Mean Bond Return	0.827	2.632	2.647

Table 1: The equity premium and the average stock and bond returns in the models with stochastic depreciation or adjustment costs, and in the base model, with the fixed benefit policy, and with different specification of the borrowing costs.

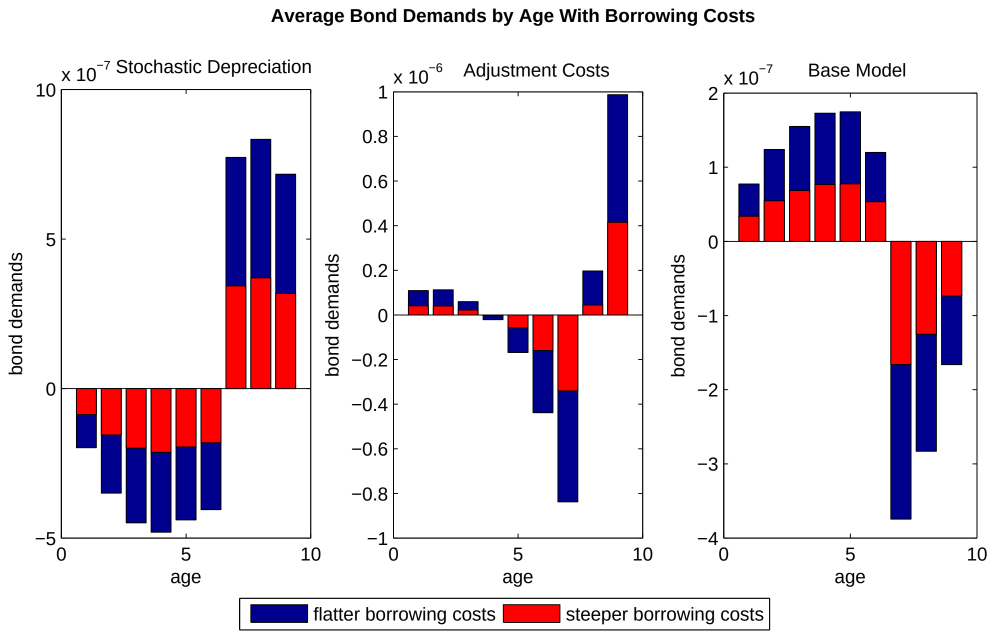

Figure 5: The average bond demands by age in the model with stochastic depreciation, the fixed benefit policy, and the borrowing costs.

4.3 Bond Demands and the Equity Premium With Borrowing Costs

Just as the demand for bonds is affected by different market conditions, it is also affected by the introduction of borrowing costs. Figure 5 shows that borrowing costs limit the supply of bonds, more so when they are steeper. The two bottom panels of Table 1 show that this increases the bond price and reduces the safe rate of return, regardless of the model.

Figure 6 plots the returns and the equity premium through time. It shows that the realized equity premium fluctuates quite a bit regardless of the presence of borrowing costs, exhibiting a standard deviation of 0.8 percent.

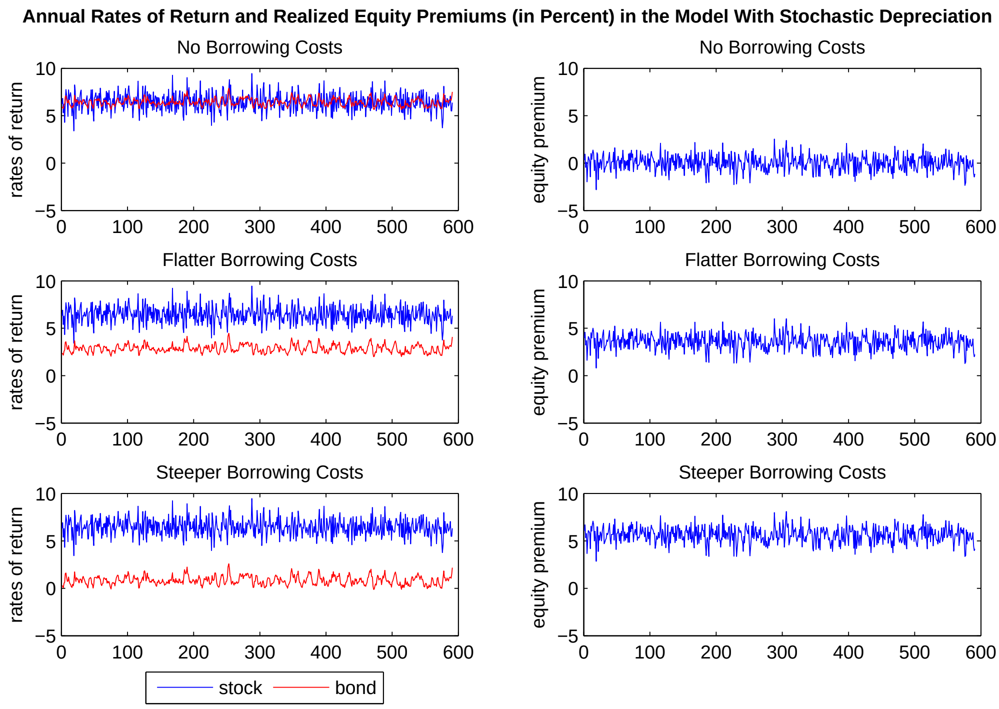

Figure 6: The annual stock and bond returns and the realized equity premiums (in percentage points) for the model with stochastic depreciation, the fixed benefit policy, with or without borrowing costs.

If the young are limited in the amount of bonds they can supply, they are also limited in the amount of stocks they can buy. This leads to a concentration of stock ownership among

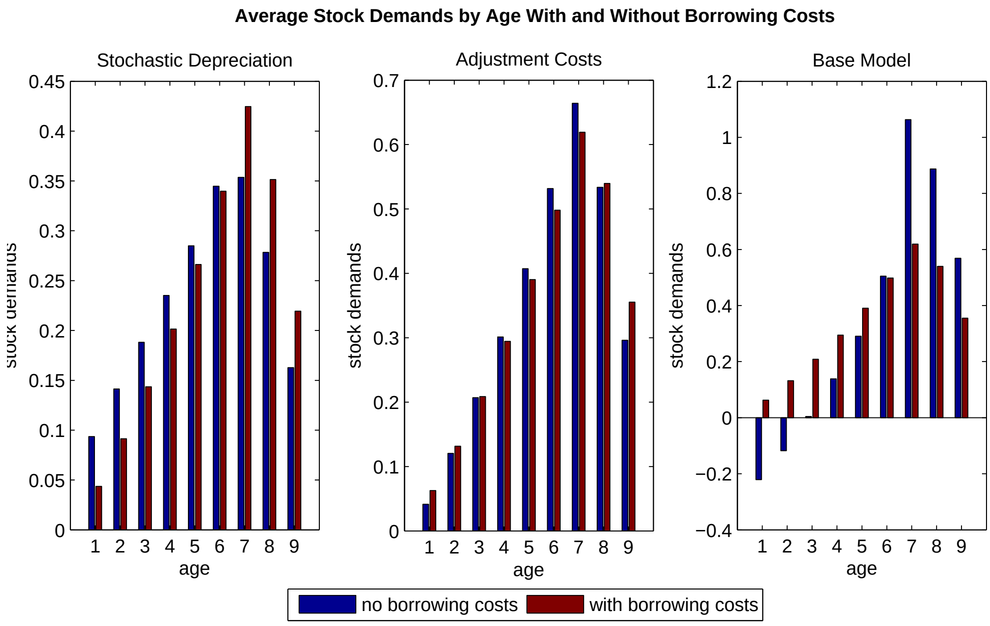

Figure 7: The average stock demands by age in the model with stochastic depreciation, the fixed benefit policy, and the borrowing costs.

the generations who would rather hold bonds—the elderly in the main model or the young in the base model. Consequently, stocks become priced by generations who desire them less, as illustrated in Figure 7. This drives the stock price down and the stock return up, as in Constantinides, Donaldson, and Mehra (2002). However, quantitatively speaking, the effect on the stock returns is small. For example, in the model with stochastic depreciation, the average stock return increases from 6.439 percent to 6.441 percent with the addition of borrowing costs. The main mechanism by which the equity premium enlarges is the drop in the risk-free rate.

4.4 Volatility

A realistic calibration of the shocks suffices for a large equity premium to emerge via increasing borrowing costs. But it does not suffice for the model to match the high stock market volatility observed in the data. For example, in the model with stochastic depreciation, the standard deviation of stock returns is 0.900 percent annually, which is an order of magnitude lower than that in the data. Consequently, the annual Sharpe ratio, 6.237, is an order of magnitude too high. A standard way to increase the volatility of stock returns is to increase the volatility of capital depreciation shocks. This also increases the equity premium, since it exposes the elderly to more risks which further boosts their demand for bonds. On net, the Sharpe ratio falls. For example, increasing the volatility of capital depreciation shocks by a factor of 1.3 yields a 25 percent increase in the volatility of stock returns, a 4 percent increase in the equity premium, and a 20 percent decrease in the Sharpe ratio. One could produce a realistic stock market volatility via unrealistic, extremely variable capital depreciation shocks, as in Krueger and Kubler (2006). This paper's solution algorithm does not seem to converge in that case.

5 Sensitivity Analysis

5.1 Alternative Policy

The previous results were obtained with the fixed benefit policy in place. This section shows that an alterative policy—the proportional tax policy—does not affect the equity premium results.

Figure 8 plots the average bond shares and bond demands by age in the model with adjustment costs and without borrowing costs under each of the two policies. It shows that the choice of the policy matters for bond holdings. For example, the direction of bond positions of the young flips when the policy is changed—they are long bonds under the fixed benefit policy and short bonds under the proportional tax policy.

This pattern is intuitive since the two policies have different implications for the distribution of risk across generations. Under the fixed benefit policy, the workers pay a higher proportion of their wages when times are bad, which increases their demand for bonds. On the other hand, when taxes are proportional, the amount of transfers is higher when times

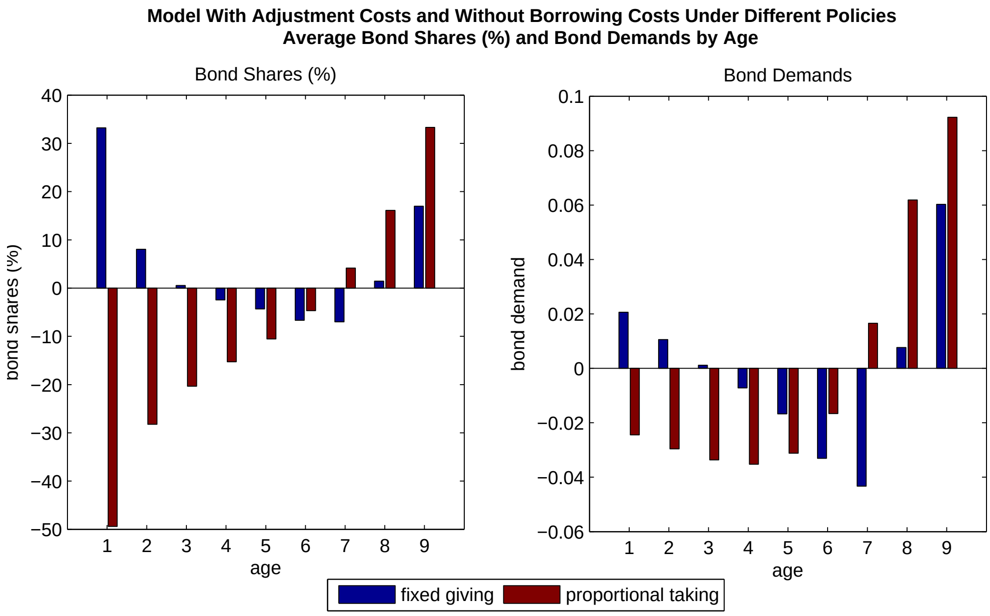

Figure 8: Average bond demands by age in the model with adjustment costs, fixed giving or proportional taking policy, and without borrowing costs.

Statistic (Annual, in Percent)	Stochastic Depreciation	Adjustment Costs	Base Model
	No Borrowing Costs		
Equity Premium	0.027	0.011	0.001
Mean Stock Return	8.197	9.306	9.220
Mean Bond Return	8.170	9.295	9.219
	Flatter Borrowing Costs		
Equity Premium	3.275	3.081	3.069
Mean Stock Return	8.207	9.308	9.306
Mean Bond Return	4.932	6.227	6.237
	Steeper Borrowing Costs		
Equity Premium	5.107	4.800	4.785
Mean Stock Return	8.207	9.308	9.306
Mean Bond Return	3.100	4.508	4.521

Table 2: Equity premium and average stock and bond returns in the models with stochastic depreciation or adjustment costs, and in the base model, with proportional taking policy, with and without borrowing costs.

are good. This reduces the workers' risks and leads them to supply more bonds. However, comparing Tables 1 and 2 shows that, quantitatively speaking, the sensitivity of the equity premium to the policy choice is negligible.

5.2 Borrowing Costs on a Subset of Generations

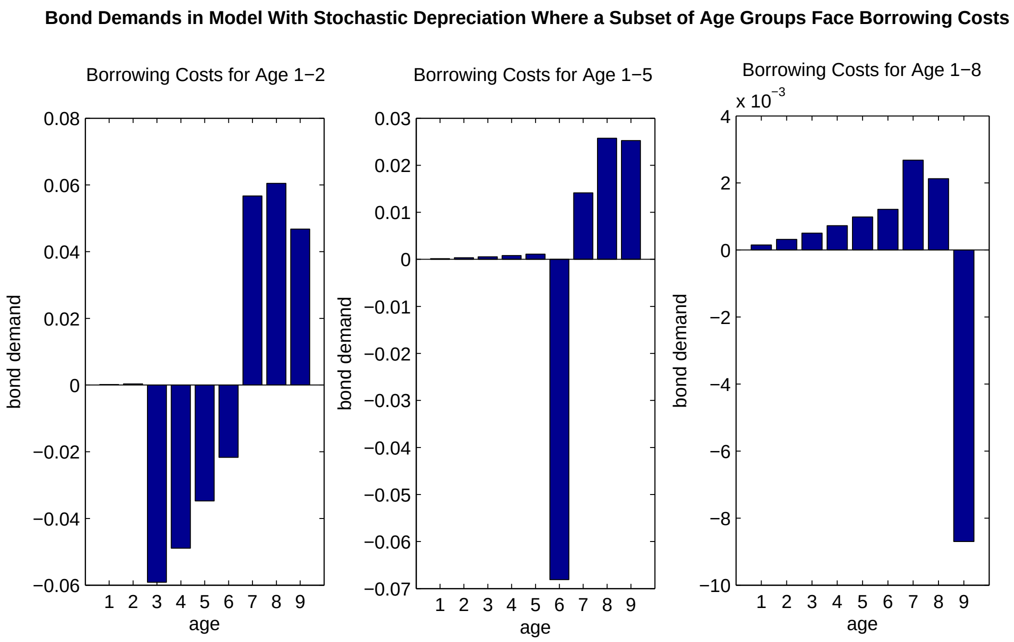

Figure 9: Bond demands by age in the model with stochastic depreciation, fixed giving policy, and steeper borrowing costs imposed on a subset of age groups.

Recall that the previous results were obtained with borrowing costs imposed on all generations. Figure 9 plots average bond demands by age for the model with stochastic depreciation where borrowing costs are imposed on different subsets of generations. It shows that if the costs are imposed only on the young, the middle-aged supply all the bonds that the old demand. And if the costs are imposed on everyone but the oldest generation, that generation becomes the supplier. Of course, the middle-aged, and especially the elderly, are not the natural suppliers of bonds in this model. Hence, the gross supply of bonds is more limited when they are the suppliers. However, it is not limited enough to significantly lower the risk-free rate and yield a sizable equity premium.

For example, in the model without borrowing costs, the gross supply of bonds is 0.201 on average. This value goes down to 0.164 if only the young face borrowing costs, further down to 0.068 if both the young and the middle-aged face the costs, and still further down to 0.0087 if everyone except the oldest faces the costs. The unlikely suppliers get compensated by a somewhat lower risk-free rate (higher bond price)—the average risk-free rate declines from 6.418 percent without borrowing costs to 6.402 percent when everyone except for the oldest faces the costs. However, at 0.038 percent, the equity premium remains at least an order of magnitude too small.

These findings are in contrast with Constantinides, Donaldson, and Mehra (2002), where the equity premium emerges with borrowing constraints imposed only on the young.

6 Accuracy of Solutions

	Fixed Benefit Policy			Proportional Tax Policy		
	Min	Mean	Max	Min	Mean	Max
	No Borrowing Costs					
Stochastic Depreciation	0.003	0.007	0.008	0.005	0.006	0.009
Adjustment Costs	0.002	0.002	0.004	0.005	0.006	0.007
Base Model	0.001	0.004	0.023	0.001	0.010	0.042
	Flatter Borrowing Costs					
Stochastic Depreciation	0.004	0.007	0.010	0.005	0.007	0.010
Adjustment Costs	0.002	0.004	0.005	0.004	0.006	0.007
Base Model	0.001	0.002	0.008	0.001	0.004	0.023
	Steeper Borrowing Costs					
Stochastic Depreciation	0.004	0.007	0.010	0.004	0.006	0.010
Adjustment Costs	0.003	0.003	0.005	0.003	0.005	0.006
Base Model	0.001	0.002	0.007	0.001	0.004	0.023
Borr. Costs on First 2	0.004	0.008	0.010	0.004	0.008	0.010
Borr. Costs on First 5	0.004	0.006	0.009	0.004	0.006	0.009
Borr. Costs on First 8	0.004	0.007	0.011	0.004	0.007	0.011
More Volatile Depreciation	0.005	0.009	0.014	0.005	0.009	0.014

Table 3: Minimum, mean, and maximum across generations of the average, across time, of the absolute value of the generation-specific, out-of-sample deviations from the perfect satisfaction of Euler equations.

A satisfactory solution requires the generation-specific Euler equations (15) hold out of sample, i.e., on a set draws for the shocks not used to compute the equilibrium decision rules. Hence, to test the accuracy of solutions, for each model considered we draw a fresh sequence of z's and δ's that is 60 times longer than the 640-period sequence used in the original simulation. We then simulate the model forward on the new path of shocks, using the original asset demand functions, θ_g, and clearing the bond market in each period.[4] We calculate the out-of-sample percentage deviations from full satisfaction of the Euler equations, for each period in the newly simulated time path and for each generation $g \in 1, \ldots, G-1$.[5] Finally, we compute the average, across time, of the absolute value of the deviations from Euler equations for each generation. Table 3 reports the summary statistics, across generations, of their average absolute deviations from Euler equations for each model considered.

The largest deviation—4 percentage points—is observed in the base model without borrowing costs and with proportional tax policy. Most other deviations are much less than 1 percent.

7 Conclusion

Simulating a sizable equity premium in macroeconomic models has proved difficult, hence the "equity premium puzzle". To explain the puzzle, economists had to apply a lot of machinery. This paper shows that a sizable equity premium can easily be obtained in a standard, multi-period OLG setting. This is demonstrated in a ten-period GE OLG model with aggregate uncertainty. The base model is quite simple: it features isoelastic preferences with modest risk aversion, Cobb-Douglas production technology, and realistic TFP shocks. On the fiscal side it includes government consumption, as well as an intergenerational redistribution policy which can be relabeled as government debt. The critical extra ingredient needed to produce

[4] For details of the solution method, including the bond market clearing algorithm, see the Appendix.
[5] The out-of-sample test does not apply to (16) since the inner loop is rerun, i.e. (16) will hold by construction.

a sizable equity premium is the increasing cost of supplying bonds, i.e. of borrowing. These costs, implemented via a smoothing function proposed by Chen and Mangasarian (1996) are smooth but essentially zero when bond holdings are positive, and are rising as bond holdings become negative. A sizable equity premium emerges immediately with the aforementioned features, but producing a pattern of increasing bond demands by age requires extra elements, namely modest stochastic depreciation or capital adjustment costs. The findings are robust to policy changes.

The model builds on Hasanhodzic and Kotlikoff (2013). As in that paper, it is solved using Marcet (1988) and Judd, Maliar, and Maliar (2009, 2011) to overcome the curse of dimensionality.

The results echo, but also differ from those of Constantinides, Donaldson, and Mehra (2002), who use a three-period, partial equilibrium OLG model with pure exchange. In their model, hard borrowing constraints on the young suffice to limit the supply of bonds and yield a large equity premium. Here, only when all generations are subject to borrowing costs is the supply of bonds limited enough for the equity premium to emerge.

When the shocks are realistically calibrated, the model does not match the stock market volatility and, consequently, the Sharpe ratio observed in the data. A standard way to match the volatility of stock returns is to assume unrealistic, extremely variable shocks, as done, e.g., in Krueger and Kubler (2006). However, this paper's solution algorithm does not seem to converge in that case. A different route to matching the stock market volatility might be to assume alternative preferences. This is left for future work.

A Computational Appendix

At the high level, the algorithm closely follows that of Hasanhodzic and Kotlikoff (2013). However, the low-level execution presents different issues because getting the model to converge is more challenging in the presence of borrowing costs, adjustment costs, and stochastic depreciation. For completeness, and to highlight where different equilibrium conditions are used, the high-level structure is outlined below.

The algorithm consists of an inner loop and an outer loop. The outer loop solves for the asset demand functions of each age group by porting Judd, Maliar, and Maliar's (2009, 2011) generalized stochastic simulation algorithm (GSSA) to the OLG setting. It starts by making an initial guess of generation-specific asset demand functions θ_g as polynomials in the state variables. Next it draws a path of the shocks for T periods and runs the model forward over those periods using the guessed asset demand functions to compute the state variables in each period. Then, for each age group, g, it evaluates the Euler equation (15) to determine what age group g's asset demand should be in each period t. Finally, it regresses these time series of generation-specific asset demands on the state variables, and uses the regression estimates to update the corresponding polynomial coefficients. It repeats these steps using the same path of shocks until asset demand functions converge.

The inner loop is the extension of GSSA by Hasanhodzic and Kotlikoff (2013) that allows for the bond market. It consists of a binary search algorithm which determines the risk-free rate $\bar{r}$ that satisfies (13). In this binary search, the evaluation of the net bond demand is achieved by using another binary search to determine the unique bond shares that satisfy the first order conditions (16).

The following is the step-by-step description.

Initialization:

- Set $\bar{z} = 1$, $\bar{\delta} = \mu_\delta$, and solve for the nonstochastic steady state asset demands of each age group without bond, $\bar{s} = (\bar{s}_1, \ldots, \bar{s}_{G-1})$. Let $(s_0, z_0, \delta_0) = (\bar{s}, \bar{z}, \bar{\delta})$ be the starting point of the simulation.

- Approximate $G-1$ asset demand functions by polynomials in the state variables: $\theta_1(s, z, \delta) = \phi_1(s, z, \delta; b_1), \ldots, \theta_{G-1}(s, z, \delta) = \phi_{G-1}(s, z, \delta; b_{G-1})$, where $b_1, \ldots b_{G-1}$ are polynomial coefficients. We use degree 1 polynomials. To start iterations, we use the following initial guess for the coefficients: $b_1 = (0, 0.9, 0, \ldots, 0, 0.1\bar{s}_1, 0), \ldots, b_{G-1} = (0, 0, \ldots, 0, 0.9, 0.1\bar{s}_{G-1}, 0)$. Note that for all $g \in \{1, \ldots, G-1\}$, the initial b_g is such that $\bar{s}_g = \phi_g(\bar{s}, \bar{z}, \bar{\delta}; b_g)$.

Outer loop:

- Take draws of the path of z's and δ's for T years. We set T to 640.
- Simulate the model forward for $t = 0, \ldots, T$. More precisely, at time t, for each age group g, calculate its asset demand $\theta_g^{(p)}$ given the current guess for the coefficients $b_g^{(p)}$, where the subscript (p) denotes the current iteration of the outer loop. I.e., $\theta_{g,t}^{(p)}$ equals the inner product of the vector $(1, s_t, z_t, \delta_t)$ with the vector of coefficients $b_g^{(p)}$, where $s_t = (\theta_1^{(p)}(s_{t-1}, z_{t-1}, \delta_{t-1}), \ldots, \theta_{G-1}^{(p)}(s_{t-1}, z_{t-1}, \delta_{t-1}))$. Then the state at time $t+1$ and iteration p is given by $(s_{t+1}, z_{t+1}, \delta_{t+1}) = (\theta_1^{(p)}(s_t, z_t, \delta_t), \ldots, \theta_{G-1}^{(p)}(s_t, z_t, \delta_t), z_{t+1}, \delta_{t+1})$, where z_{t+1} given z_t is determined by (6).
- Inner loop:
  - Use binary search to solve (13) for $\bar{r}_t$, for all $t = 0, \ldots, T$. To start, make an (arbitrary) initial guess for the value of $\bar{r}_t$.
  - For all $t = 0, \ldots, T$, given $\bar{r}_t$, for all $g = 1, \ldots, G-1$, solve (16) for $\alpha_{g-1,t}$ using another binary search (evaluate the expectation in (16) using Gaussian quadrature).
  - Use $\alpha_{g-1,t}$ found above for all g and for all t to calculate (13) and update $\bar{r}_t$ for all t.

- Note that for each age group g and each state (s_t, z_t, δ_t), $t = 1, \dots, T$, (15) implies

$$
\begin{aligned}
\theta_g(s_t, z_t, \delta_t) = \beta \mathrm{E}_z \big[\theta_g(s_t, z_t, \delta_t) \big(1 + r(s_{t+1}, z_{t+1}, \delta_{t+1}) \\
+ \alpha_g(s_t, z_t, \delta_t)(\bar{r}(s_t, z_t, \delta_t) - r(s_{t+1}, z_{t+1}, \delta_{t+1})) \\
- f(\alpha_g(s_t, z_t, \delta_t)) \big) \frac{u'(c_{g+1}(s_{t+1}, z_{t+1}, \delta_{t+1}))}{u'(c_g(s_t, z_t, \delta_t))} \big]
\end{aligned}
\tag{A.1}
$$

for equilibrium asset demands θ_g. Denote the right-hand-side of (A.1) by $y_g(s_t, z_t, \delta_t)$ and evaluate the expectation using Gaussian quadrature.

- For each age group g, regress $y_g(s_t, z_t, \delta_t)$ on (s_t, z_t, δ_t) and a constant term using regularized least squares with Trikhonov regularization (see Judd, Maliar, and Maliar, 2011 for details). Denote the estimated regression coefficients by $\hat{b}_g^{(p)}$.

- Check for convergence: If

$$
\frac{1}{G-1} \sum_{g=1}^{G-1} \frac{1}{T} \sum_{t=1}^{T} \left| \frac{\theta_g^{(p-1)}(s_t, z_t, \delta_t) - \theta_g^{(p)}(s_t, z_t, \delta_t)}{\theta_g^{(p-1)}(s_t, z_t, \delta_t)} \right| < \epsilon,
$$

end. Otherwise, for each age group g update the coefficients as $b_g^{(p+1)} = (1-\xi) b_g^{(p)} + \xi \hat{b}_g^{(p)}$, for $\xi = 0.01$, and return to the beginning of the outer loop.

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